THEORETICAL & COMPUTATIONAL CONSIDERATIONS OF STURM-LIOUVILLE SYSTEMS.

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SINCERELY DEDICATED
TO
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EARNEST GRATITUDE TOWARDS MY GURUJI
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ABSTRACT

A common exercise encountered while studying analysis is to decompose a certain object in terms of a given basis. In particular, we focus on the representation of a given function $f \in L^2[a,b]$ by an infinite series involving functions from a certain basis of the function space. The idea of representations by series are often encountered while modelling and solving boundary value problems in ordinary and partial differential equations. To begin with, we discuss representations by Fourier series, which are certain types of infinite series constituting multi-angle sinusoidals. In Chapter 1, we present some theory on Fourier series, discuss its convergence and also go through some important numerical aspects of Fourier series approximations.

Once it is established that the set of multi-angle sinusoidals, i.e.,

$$\mathcal{F} = \{1, \sin nx, \cos nx \mid \text{for } n \text{ running over } \mathbb{N}\},$$

building blocks of a Fourier series, forms a basis of $L^2[-\pi,\pi]$, we see Fourier series from the perspective of regular SL systems. Note that, $\mathcal{F}$ constitutes eigenfunctions of a regular SL system. Hence, gradually in Chapter 2 we study the theory of Sturm-Liouville systems, in an attempt to generalise the concept of a Fourier series to an SL series. While the beginning of Chapter 2 mainly discusses types of SL systems and some results on eigenfunctions of SL systems, the main crux of it is to present the Oscillation Theory.

The discussion is then naturally carried forward in Chapter 3, where first of all existence of a sequence of eigenfunctions of any regular SL system is presented. Chapter 3 then discusses some theory on the asymptotic behaviour of eigenfunctions and distributions of eigenvalues of regular SL systems, eventually leading to establish the completeness of the set of eigenfunctions in the function space $L^2_{\rho(x)}[a,b]$.

Next, in Chapter 4, we broaden our persepective of looking at SL series approximations and hence consider concepts from approximation theory. For instance, one can formulate and present a suitable approximation problem keeping in mind the associated geometrical properties of a Fourier series approximation. Similarly from an application point of view, there exists important approximation problems which are solved by Chebyshev Polynomials. Finally, we present numerical methods for performing Chebyshev approximations efficiently. At a glimpse, unlike the numerical experiments on Fourier series approximations where the coefficients were computed using numerical integration, we discuss discrete computation of coefficients in Chebyshev series and fast numerical evaluation of Chebyshev series.

LIST OF SYMBOLS & ABBREVIATIONS

$\mathbb{R}$	the set of all real numbers
$\mathbb{N}$	the set of all natural numbers
$\mathbb{R}_{>0}$	the set of all positive real numbers
$\mathbb{Z}$	the set of all integers
$\mathbb{Z}_{>0}$ or $\mathbb{N}$	the set of all positive integers
$\mathbb{Z}_{\geq 0}$ or $\mathbb{N}_0$	the set of all non-negative integers
$\mathbb{T}$	the interval $[-\pi, \pi]$
I	the interval $[-1, 1]$
$\mathcal{P}_n$	the space of all algebraic polynomials upto degree n
$\mathcal{T}_n$	the space of all trigonometric polynomials of degree n
$C^k_{2\pi}(-\pi, \pi)$	$\{f : (-\pi, \pi) \to \mathbb{R} \mid f$ is k-times continuously differentiable and 2π periodic$\}$
$C^k(a, b)$	$\{f : (a, b) \to \mathbb{R} \mid f$ is k-times continuously differentiable$\}$
$L^2(a, b)$	$\{f : (a, b) \to \mathbb{R} \mid f$ is Lebesgue measurable, $\|\|f\|\|_{L^2(a,b)} < \infty\}$
c^f_n	the n^{th} term truncated Chebyshev series for a function f
s^f_n	the n^{th} term truncated Fourier series (or an SL series) for a function f

DE	differential equation
ODE	ordinary differential equation
DO	differential operator
SL	*Sturm-Liouville*

LIST OF FIGURES

LIST OF TABLES

TABLE OF CONTENTS

CHAPTER 1: FOURIER SERIES

1.1 Introduction

The genesis of Fourier analysis, dating back to the 19th century, marks a watershed moment in scientific inquiry. Pioneered by Jean-Baptiste Joseph Fourier in his quest to understand heat conduction[14], this concept revealed that complicated functions involved in certain periodic physical phenomena could be deconstructed into simpler sinusoidal components. Today, Fourier analysis stands as an enduring pillar of scientific investigation, serving as a vital tool in deciphering intricate signals and waves across a diverse array of disciplines[24],[6],[10]. In this chapter, we discuss some elementary results which are naturally encountered while developing the foundational concepts of Fourier analysis.

Definition 1.1.1. *For a given function f, we can associate it with an infinite series of sinusoids. Whenever*

$$\begin{aligned} f(x) = A_0 + &a_1\cos(x) + a_2\cos(2x) + \cdots \\ &+ b_1\sin(x) + b_2\sin(2x) + \cdots \end{aligned} \tag{1}$$

such a series is called a ***Fourier Series****.*

Under some hypotheses of considerable generality on the function f, this sinusoidal infinite series can represent f. We define a constant p to be a period for a function $\phi(x)$, if $\phi(x+p) = \phi(x)$. Note that, our definition for a *period* remains the same even though p may not be the smallest value for which a relation of this sort is satisfied.

1.2 Orthogonality of Sines and Cosines

Firstly, for non-zero integer n, we observe the following definite integrals:

$$\int_{-\pi}^{\pi} \cos(nx) = 0, \qquad \int_{-\pi}^{\pi} \sin(nx) = 0 \tag{2}$$

Here, the second equation holds true even if $n = 0$, while in the second equation the right-hand side value becomes 2π for $n = 0$.

Throughout the discussions ahead, we consider p and q to be non-negative integers. Using the relation $\cos(px)\cdot\cos(qx) = \frac{1}{2}\cos((p-q)x) + \frac{1}{2}\cos((p+q)x)$ and the relations at (2), we get:

$$\int_{-\pi}^{\pi} \cos(px)\cdot\cos(qx)dx = 0, \quad \text{when } p \neq q. \tag{3}$$

Even if $p = q \neq 0$, the integral of $\cos((p+q)x)$ over the period interval is zero, and the other term gives us:

$$\int_{-\pi}^{\pi} \cos^2(px) = \pi.$$

Similarly, the following identities:

$$\sin(px) \cdot \sin(qx) = \frac{1}{2}\cos((p-q)x) - \frac{1}{2}\cos((p+q)x),$$
$$\sin(px) \cdot \cos(qx) = \frac{1}{2}\sin((p-q)x) - \frac{1}{2}\sin((p+q)x),$$

eventually gives us,

$$\int_{-\pi}^{\pi} \sin(px) \cdot \sin(qx)dx = 0, \quad \text{when } p \neq q, \tag{4}$$
$$\int_{-\pi}^{\pi} \sin^2(px) = \pi \quad \text{when } p \neq 0,$$
$$\int_{-\pi}^{\pi} \sin(px) \cdot \cos(qx)dx = 0. \tag{5}$$

Definition 1.2.1. *In general, two functions $u(x)$ and $v(x)$ are said to be* ***orthogonal*** *to each other over an interval (a, b) if*

$$\int_a^b u(x)v(x)dx = 0.$$

The vanishing of the integrals in (3), (4) and (5) can be expressed by stating that any two of the functions $1,\ \cos(x),\ \cos(2x),\ \cos(3x),\ \cdots,\ \sin(x),\ \sin(2x),\ \sin(3x),\ \cdots$ are *orthogonal* to each other over the interval $(-\pi, \pi)$.

The integral involved in the definition of orthogonal functions given above is well-defined when u and v are piece-wise continuous on the interval $a < x < b$. So, one can define an inner product of functions f and g in $C_p(a, b)$ [10], the function space of all piece-wise continuous functions on the interval $a < x < b$ as:

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx.$$

Here, it is easy to verify *positive definiteness*, *symmetricity* and *bilinearity* for this inner product. Now considering this inner product defined above, one can say that functions f and g are orthogonal to each other if their inner product is zero.

1.3 Determination of the coefficients

For the sake of formal calculations we integrate f in (1), by integrating the series term-by-term. So, the integration of (1) using (2) gives us:

$$\int_{-\pi}^{\pi} f(x)dx = 2\pi A_0$$
$$\implies A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.$$

Now, in order to determine a_k for non-zero k, let identity (1) be multiplied through by $\cos(kx)$ and the resulting expression for $f(x)\cos(kx)$ be integrated from $-\pi$ to π, still under the assumption of linearity of integration. When doing so, again, each integral on the right side of the expression reduces to zero, as a consequence of (2), (3) and (5), except the term with $\cos^2(kx)$ as the integrand and it is found that :

$$\int_{-\pi}^{\pi} f(x)\cos(kx)dx = a_k \int_{-\pi}^{\pi} \cos^2(kx)dx = \pi a_k$$
$$\implies a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx. \tag{6}$$

Similarly, we will have :

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx. \tag{7}$$

Here, the formula for a_k does not reduce to that for A_0 if k is equal to 0. However, if $2A_0$ is denoted by a_0, then this a_0 is given by (6) with $k = 0$. Thus, from now on, a Fourier series will regularly be written in the form,

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k\cos(kx) + b_k\sin(kx)], \tag{8}$$

with all coefficients, including a_0, given by (6) and (7).

1.3.1 Dependence on the period

If $F(y)$ is a function of the variable y with period $2p$, where p is an arbitrary positive number and let $F(y)$ as a function of x be denoted by $f(x)$. Basically, we obtain $F(y)$ by performing the following change of variable in $f(x)$:

$$x = \frac{\pi y}{p}, \qquad y = \frac{px}{\pi}$$

Then $f(x)$ has the period 2π in terms of x. If $f(x)$ is represented by a series of the form (8), this constitutes a representation of $F(y)$ in the form :

$$F(y) = \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k\cos\left(\frac{k\pi y}{p}\right) + b_k\sin\left(\frac{k\pi y}{p}\right)],$$

and the formulas given at (6) and (7) for the coefficients become :

$$a_k = \frac{1}{p}\int_{-p}^{p} F(y)\cos\left(\frac{k\pi y}{p}\right)dy, \qquad b_k = \frac{1}{p}\int_{-p}^{p} F(y)\sin\left(\frac{k\pi y}{p}\right)dy.$$

Also, if one considers $\phi(x)$ to be a periodic function with period 2π, then by a change of variable, we have :

$$\int_{a}^{a+2\pi} \phi(x)dx = \int_{b}^{b+2\pi} \phi(x)dx \qquad \forall\ a, b \in \mathbb{R}.$$

To put it formally, if a periodic function is integrated over a period interval, this interval can be replaced by any other interval of the same length without changing the value of the integral.

Now we can summarize the discussions made above with respect to a Fourier series. In context of the Fourier series of a function of period 2π, the integrals may be written as extended over the interval $(0, 2\pi)$ instead of $(-\pi, \pi)$, and also the use of still other period intervals is equally permissible. Moreover, it is now clear that, a Fourier series representation of the form given at (1) is possible not only for a *2π-periodic* function but for any periodic function with period $2p$.

1.4 Series of Sines and Cosines

We start this section by stating a standard result from elementary analysis, that if $\phi(x)$ is an even function, that is, $\phi(x) = \phi(-x)$, and if it is integrated over any interval $(-a, a)$, then we have :

$$\int_{-a}^{a} \phi(x)dx = 2\int_{0}^{a} \phi(x)dx.$$

Similarly, if $\phi(x)$ is odd, that is, if $\phi(-x) = -\phi(x)$, then :

$$\int_{-a}^{a} \phi(x)dx = 0.$$

Using this result above, we consider the following two cases in any interval of period 2π.

1.4.1 Case A : $f(x)$ is an even function

In this case, when $f(x)$ is even, the function $f(x)\cos(kx)$ is also even and the function $f(x)\sin(kx)$ is odd, for each value of k. When the coefficients in the Fourier series for $f(x)$ are defined by (6) and (7), we have :

$$a_k = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(kx)dx, \qquad b_k = 0. \tag{9}$$

Thus, the Fourier series for an even function contains only the cosine terms and the coefficients are given by (9).

1.4.2 Case B : $f(x)$ is an odd function

Now, the products $f(x)\cos(kx)$ and $f(x)\sin(kx)$ are odd and even functions respectively. The Fourier series contains only sine terms, and the coefficients being given by :

$$a_k = 0, \qquad b_k = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(kx)dx. \tag{10}$$

Now from the discussion done through the two cases, we must notice that the formulas (9) and (10) in themselves involve the values of the function $f(x)$ only in the interval $(0, \pi)$.

Thus, any function which is integrable from 0 to π, that is, $f \in L^1[0, \pi]$ can be formally represented in that interval, without any assumptions in advanced that whether it is even or odd or periodic or defined elsewhere at all, by a series of cosines with coefficients (9) and alternatively by a series of sines with coefficients (10).

1.5 Magnitude of coefficients under special hypotheses

Let $f(x)$ be a function of period 2π, which has a continuous first derivative for all values of x. In the integral defining the Fourier coefficients a_k (6), integration-by-parts gives us

$$\begin{aligned}\pi a_k &= \int_{-\pi}^{\pi} f(x)\cos(kx)dx \\ &= \underbrace{[(\frac{1}{k})f(x)\sin(kx)]\Big|_{-\pi}^{\pi}}_{\text{this part vanishes}} - \frac{1}{k}\int_{-\pi}^{\pi} f'(x)\sin(kx)dx.\end{aligned}$$

By using Extreme Value Theorem, if M_1 is the maximum of $|f'(x)|$ then,

$$\begin{aligned}\Big|\int_{-\pi}^{\pi} f'(x)\sin(kx)dx\Big| &\le \int_{-\pi}^{\pi} |f'(x)\sin(kx)|dx \\ &\le \int_{-\pi}^{\pi} M_1 dx = 2\pi M_1.\end{aligned}$$

Thus, we have $|a_k| \le \frac{2M_1}{k}$. Similarly, we can deduce that $|b_k| \le \frac{2M_1}{k}$. A major takeaway from the discussion above is that the Fourier coefficients approach zero as the factor k approaches infinity.

The above conclusion for the coefficients does not provide us an anticipated conclusion as the series $\Sigma\frac{1}{k}$ diverges as k approaches infinity. Note that here, the *Weierstrass M-Test*[1],[23] is under usage as a sufficient condition for a series to converge. Now, we assume that $f(x)$ has a continuous second derivative, with M_2 being the maximum of $|f''(x)|$. Note that f is till periodic with period 2π. Analogous to the calculations done earlier in this section, now by doing two successive integration-by-parts, we get :

$$\begin{aligned}\pi a_k = \int_{-\pi}^{\pi} f(x)\cos(kx)dx &= -\frac{1}{k}\int_{-\pi}^{\pi} f'(x)\sin(kx)dx \\ &= \underbrace{[\frac{1}{k^2}f'(x)\cos(kx)]\Big|_{-\pi}^{\pi}}_{\text{the terms cancel out each other}} - \frac{1}{k^2}\int_{-\pi}^{\pi} f''(x)\cos(kx)dx\end{aligned}$$

Here we get, $|a_k| \le \frac{2M_2}{k^2}$, and similarly $|b_k| \le \frac{2M_2}{k^2}$.

The conclusion above can lead us as :

$$\begin{aligned}
|a_k\cos(kx)+b_k\sin(kx)| &\le |a_k||\cos(kx)|+|b_k||\sin(kx)| \\
&\le |a_k|+|b_k| \\
&\le \frac{2M_2}{k^2}+\frac{2M_2}{k^2} \\
&\le \frac{4M_2}{k^2}.
\end{aligned}$$

Now due to the series $\Sigma\frac{1}{k^2}$ being convergent, the term in the right-hand-side of the inequality is the general term of a convergent series. Thus, if f has continuous derivative of higher order, then the discussion ahead can be used to conclude that its Fourier coefficients tend to zero.

Now let us increase the generality of the class of functions to be considered. Let $f(x)$ be continuous and of period 2π. We further assume that the interval $(-\pi,\pi)$ can be divided into a finite number of sub-intervals, such that in each of which $f(x)$ is linear. Thus, the graph of $f(x)$ in any one period interval is then made up of a finite number of straight line segments of finite slope joined end-to-end. Such a *function will be called a* ***Broken Line Function*** *or piece-wise linear function.* Figure(1) depicts the graph of a typical *Broken Line Function.*

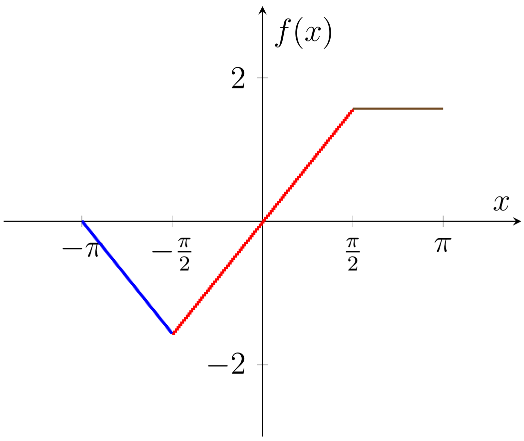


Figure 1: Graph of a typical Broken Line Function

In our context, let the abscissas of the corners in the interior of $(-\pi,\pi)$ be $x_1,x_2,\cdots,x_{m-1}$ and for the sake of uniformity of notation, let $x_0=-\pi$ and $x_m=\pi$. Further, let λ_j be the constant value of $f'(x)$ in the interval (x_{j-1},x_j), and

$$\lambda=\underset{j=1,2,\cdots,m.}{max}\{|\lambda_j|\}.$$

Now for the j^{th} sub-interval, using integration by-parts we get :

$$\begin{aligned}
\int_{x_{j-1}}^{x_j} f(x)\cos(kx)dx &= [\frac{1}{k}f(x)\sin(kx)]\Big|_{x_{j-1}}^{x_j}-\frac{1}{k}\int_{x_{j-1}}^{x_j}\lambda_j\sin(kx)dx \\
&= \underbrace{\frac{1}{k}[f(x_j)\sin(kx_j)-f(x_{j-1})\sin(kx_{j-1})]}_{\text{term A}}+\underbrace{\frac{\lambda_j}{k^2}[\cos(kx_j)-\cos(kx_{j-1})]}_{\text{term B}}.
\end{aligned}$$

So, when a summation over $m-$intervals is performed, we get :

$$\begin{aligned}\sum term\ A &= \frac{1}{k}\sum_{j=1}^{m}\Big[f(x_j)\sin(kx_j) - f(x_{j-1})\sin(kx_{j-1})\Big]\\ &= \frac{1}{k}[f(\pi)\sin(k\pi) - f(-\pi)\sin(-k\pi)]\\ &= 0.\end{aligned}$$

Also, since we have $\Big|(\frac{\lambda_j}{k^2})[\cos(kx_j) - \cos(kx_{j-1})]\Big| \leq \Big|\frac{\lambda_j}{k^2}\Big|\Big[|\cos(x_j)| + |\cos(x_{j-1})|\Big] \leq 2\lambda/k^2$, as a consequence we have :

$$\begin{aligned}\Big|\sum term\ B\Big| &= \Big|\sum_{j=1}^{m}(\frac{\lambda_j}{k^2})[\cos(kx_j) - \cos(kx_{j-1})]\Big|\\ &\leq \frac{2\lambda}{k^2}\cdot\sum_{j=1}^{m}1\\ &\leq \frac{2\lambda}{k^2}\cdot m.\end{aligned}$$

Due to finite number of x_js, we can conclude that :

$$|a_k| = \frac{1}{\pi}\Big|\sum_{j=1}^{m}\int_{x_{j-1}}^{x_j} f(x)\cos(kx)dx\Big| \leq \frac{2m\lambda}{\pi k^2}.$$

Performing similar calculations for b_k one can get that :

$$|b_k| \leq \frac{2m\lambda}{\pi k^2}.$$

Thus following the above discussion we can conclude that the Fourier coefficients a_k and b_k of a broken-line function are such that :

$$|a_k| \leq \frac{C}{k^2}; \qquad |b_k| \leq \frac{C}{k^2}, \tag{11}$$

where $C = \frac{2m\lambda}{\pi}$, independent of k.

Thus, if f is a broken line function, then again its Fourier coefficients tend to zero.

As we know that, a necessary condition for a series to be convergent is that the terms of the series must approach zero as the index of the terms becomes arbitrarily large, we can produce the major takeaway for this section by considering the above discussions.

Lemma 1.5.1. *If f is piece-wise continuously differentiable with continuous derivative of second order, then its Fourier coefficients tend to zero. Thus, for such functions at least the necessary condition, for its Fourier series to be convergent, is satisfied.*

1.6 Riemann's Theorem on the limit of general coefficients

The primary aim of this section is to introduce the Riemann's Theorem, also known as the ***Riemann-Lebesgue Lemma***, which is a fundamental result in the theory of Fourier series. It describes the behavior of the Fourier coefficients of a piece-wise continuous or in more generality, integrable function as the index of the coefficients approaches infinity[10].

Now, let $f(x)$ be any function is integrable over the interval $(-\pi, \pi)$ with an additional constraint that $[f(x)]^2$ is also integrable over $(-\pi, \pi)$. f is not necessarily periodic or defined at all outside the mentioned interval.

Let $s_n^f(x)$ be the partial sum of the Fourier series of f through terms of the n^{th} order, that is :

$$s_n^f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \big(a_k \cos(kx) + b_k \sin(kx)\big). \tag{12}$$

Now it follows that :

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) s_n^f(x) dx &= \frac{a_0}{2}\int_{-\pi}^{\pi} f(x)dx + \sum_{k=1}^{n}\Big[a_k \int_{-\pi}^{\pi} f(x)\cos(kx)dx + b_k \int_{-\pi}^{\pi} f(x)\sin(kx)dx\Big] \\ &= \frac{\pi a_0^2}{2} + \pi \sum_{k=1}^{n} [a_k^2 + b_k^2].\end{aligned}$$

Next, we intend to expand $[s_n^f(x)]^2$ and integrate term-by-term, so :

$$[s_n^f(x)]^2 = \underbrace{\frac{a_0^2}{4}}_{I} + \underbrace{a_0 \sum_{k=1}^{n} \big(a_k \cos(kx) + b_k \sin(kx)\big)}_{II} + \underbrace{\Big(\sum_{k=1}^{n} \big(a_k \cos(kx) + b_k \sin(kx)\big)\Big)^2}_{III}$$

Hence, when we integrate $[s_n^f(x)]^2$ from $-\pi$ to π, we would be getting the sum of the following three terms in the result :

$$\int_{-\pi}^{\pi} I \ dx = \int_{-\pi}^{\pi} \frac{a_0^2}{4} dx = \frac{\pi a_0^2}{2}.$$

$$\begin{aligned}\int_{-\pi}^{\pi} II \ dx &= a_0 \int_{-\pi}^{\pi} \sum_{k=1}^{n} \big(a_k \cos(kx) + b_k \sin(kx)\big) dx \\ &= a_0 \sum_{k=1}^{n} \Big(\int_{-\pi}^{\pi} a_k \cos(kx)dx + \int_{-\pi}^{\pi} b_k \sin(kx)dx\Big) \\ &= 0.\end{aligned}$$

$$\int_{-\pi}^{\pi} III \ dx = \int_{-\pi}^{\pi} \sum_{k=1}^{n} \big(a_k \cos(kx) + b_k \sin(kx)\big)^2 dx$$

$$\int_{-\pi}^{\pi} III\ dx = \int_{-\pi}^{\pi} \Big(\sum_{k=1}^{n} a_k \cos(kx)\Big)^2 dx + \int_{-\pi}^{\pi} \Big(\sum_{k=1}^{n} b_k \sin(kx)\Big)^2 dx +$$

$$2\int_{-\pi}^{\pi} \Big(\sum_{k=1}^{n} a_k \cos(kx)\Big) \cdot \Big(\sum_{k=1}^{n} b_k \sin(kx)\Big) dx$$

$$= \int_{-\pi}^{\pi} \Big[\sum_{k=1}^{n} \big(a_k^2 \cos^2(kx)\big) + \sum_{\substack{k=1\\ j=1\\ k\neq j}}^{n} \big(a_k \cos(kx) a_j \cos(jx)\big)\Big] dx +$$

$$\int_{-\pi}^{\pi} \Big[\sum_{k=1}^{n} \big(b_k^2 \sin^2(kx)\big) + \sum_{\substack{k=1\\ j=1\\ k\neq j}}^{n} \big(b_k \sin(kx) b_j \sin(jx)\big)\Big] dx +$$

$$2\int_{-\pi}^{\pi} \Big[\sum_{j=1}^{n}\sum_{k=1}^{n} \big(a_j \cos(jx) \cdot b_k \sin(kx)\big)\Big] dx$$

$$= \big(\pi \sum_{k=1}^{n} a_k^2 + 0\big) + \big(\pi \sum_{k=1}^{n} b_k^2 + 0\big) + 0$$

$$= \pi\big(\sum_{k=1}^{n} a_k^2 + b_k^2\big).$$

Thus, we have :

$$\int_{-\pi}^{\pi} \big(s_n^f(x)\big)^2 dx = \frac{\pi a_0^2}{2} + \pi\big(\sum_{k=1}^{n} a_k^2 + b_k^2\big).$$

Consequently, combining the above conclusions we get :

$$\int_{-\pi}^{\pi} \big[f(x) - s_n^f(x)\big]^2 dx = \int_{-\pi}^{\pi} \big(f(x)\big)^2 dx - 2\int_{-\pi}^{\pi} f(x) s_n^f(x) dx + \int_{-\pi}^{\pi} \big[s_n^f(x)\big]^2 dx$$

$$= \int_{-\pi}^{\pi} \big(f(x)\big)^2 dx - \Big[\frac{\pi a_0^2}{2} + \pi\big(\sum_{k=1}^{n} a_k^2 + b_k^2\big)\Big].$$

Here, in this inequality, the integrand of the integral present in the left-hand-side is positive, so the *Bessel's Inequality* follows, that is :

$$\boxed{\frac{a_0^2}{2} + \big(\sum_{k=1}^{n} a_k^2 + b_k^2\big) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} \big(f(x)\big)^2 dx.}$$

Remark that, the last inequality is valid for all possible values of n, while the right-hand-side of the inequality in independent of n, hence it is clear that, $\sum_{k=1}^{n}(a_k^2 + b_k^2)$ is convergent. Now using the necessary condition for a series to be convergent, that is, its general term must approach zero, we get :

$$\lim_{k\to\infty} a_k = 0, \qquad \lim_{k\to\infty} b_k = 0. \tag{13}$$

Thus, from the discussion done so far in this section we can present the following lemma.

Lemma 1.6.1. ***Riemann-Lebesgue Lemma (II-form)*** *For $f \in L^2(-\pi,\pi)$, its Fourier coefficients approach zero as their index approaches infinity.*

The result in Lemma(1.6.1) also remains true without the requirement of square integrability on f, and is popularly known as the *Riemann-Lebesgue Lemma (I-form)*[24]. Further, the deductions done to achieve (13) can be presented in a different way. If $\phi(x)$ is a function, not necessarily periodic, such that both ϕ and ϕ^2 are integrable over $(-\pi,\pi)$, then we have :

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\phi(u)\cos(nu)du = 0, \qquad \lim_{n\to\infty}\int_{-\pi}^{\pi}\phi(u)\sin(nu)du = 0. \tag{14}$$

Now, in the interpretation resulting (14), if all the hypotheses are satisfied by $\phi(u)$, then they will be also satisfied by $\phi(u)\sin(u/2)$ and $\phi(u)\cos(u/2)$. Thus, substituting these products in the integrals given in (14), we get :

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\phi(u)\sin(u/2)\cos(nu)du = 0, \qquad \lim_{n\to\infty}\int_{-\pi}^{\pi}\phi(u)\cos(u/2)\sin(nu)du = 0,$$

and finally adding them we get :

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\phi(u)\sin\big((n+\frac{1}{2})u\big)du = 0. \tag{15}$$

Remark : The implications provided by the Riemann-Lebesgue Lemma also remain true without the requirement of square integrability, that is, $[f]^2$ is integrable. A direct proof for (15) for this more general case can be found at [1].

1.7 Evaluations of a sum of Cosines

Let the sum given as :

$$\mathcal{G}(v) = \frac{1}{2} + \sum_{k=1}^{n}\cos(kv),$$

be multiplied by $2\sin(v/2)$. For $k \geq 1$, let the products be evaluated by the relation :

$$2\sin(v/2)\cos(kv) = \sin\big((k+1/2)v\big) - \sin\big((k-1/2)v\big).$$

Therefore, we have :

$$\begin{aligned}
2\sin(v/2)\mathcal{G}(v) &= \sin(v/2) + \underbrace{\sum_{k=1}^{n}\Big[\sin\big((k+1/2)v\big) - \sin\big((k-1/2)v\big)\Big]}_{\text{a telescopic sum}} \\
&= \sin(v/2) + \Big[\sin\big((n+1/2)v\big) - \sin(v/2)\Big] \\
&= \sin\big((n+1/2)v\big),
\end{aligned}$$

and finally for $v \neq 0$ we get :

$$\frac{1}{2} + \cos(v) + \cos(2v) + \cos(3v) + \cdots + \cos(nv) = \frac{\sin\big((n+1/2)v\big)}{2\sin(v/2)}. \tag{16}$$

1.8 Integral formula for the partial sums of Fourier series

For the definition of $s_n^f(x)$ in (12), let the formulas for the coefficients be written with t as the variable of integration :

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)dt, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)dt.$$

Now, if we form the products $a_k\cos(kx)$ and $b_k\sin(kx)$ with the expressions above for the Fourier coefficients, then the factors $\cos(kx)$ and $\sin(kx)$ being constants with respect to the variable of integration, can be put inside the integrals and we get :

$$\begin{aligned} a_k\cos(kx) + b_k\sin(kx) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\cos(kx)dt + \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\sin(kx)dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos\Big(k(t-x)\Big)dt. \end{aligned}$$

So, $s_n^f(x)$ has another representation as :

$$\begin{aligned} s_n^f(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\Big[\frac{1}{2} + \sum_{k=1}^{n}\cos\big(k(t-x)\big)\Big]dt, \text{ which by using (16), with } v = t-x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin\big[(n+1/2)(t-x)\big]}{2\sin\big[1/2(t-x)\big]}dt. \end{aligned} \tag{17}$$

Now let us suppose that, $f(x)$ has period 2π. In (17), if we make a change of variable, with the substitution $u = t - x$, the limits of integration at first changes to $-\pi - x$ to $\pi - x$, but that would be equivalent to having the integral from $-\pi$ to π, as $f(x)$ is periodic with period 2π, with respect to u. So, we have :

$$s_n^f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\frac{\sin\big[(n+1/2)u\big]}{2\sin(u/2)}du. \tag{18}$$

1.9 Convergence at a point of continuity

By integrating (16) from $-\pi$ to π :

$$\begin{aligned} &\int_{-\pi}^{\pi}\frac{\sin\big[(n+1/2)u\big]}{2\sin(u/2)}du = \int_{-\pi}^{\pi}\frac{1}{2}du + \int_{-\pi}^{\pi}\cos(u)du + \cdots + \int_{-\pi}^{\pi}\cos(nu)du \\ \Longrightarrow &\int_{-\pi}^{\pi}\frac{\sin\big[(n+1/2)u\big]}{2\sin(u/2)}du = \pi. \end{aligned} \tag{19}$$

On multiplying both sides of (19) by $\frac{f(x)}{\pi}$, we get the following keeping in mind that $f(x)$ is constant with respect to the integrating variable u,

$$f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{\sin\big[(n+1/2)u\big]}{2\sin(u/2)}du. \tag{20}$$

Next, doing the substraction (18) - (20) we get :

$$s_n^f(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(f(x+u) - f(x)\Big)\frac{\sin\big[(n+1/2)u\big]}{2\sin(u/2)}du. \tag{21}$$

It is clear that the proof of convergence involves showing that under suitable hypothesis, the expression $[s_n^f(x) - f(x)]$ approaches 0 as n tends to infinity.

Let $f(x)$ be an integrable function of period 2π such that $[f(x)]^2$ is also integrable over an interval of period length. This condition is certainly satisfied, if f is considered to be everywhere continuous, or if it is continuous except for a finite number of finite jumps in a period.

As we aim for point-wise convergence, for now we assume that the Fourier series is convergent to the respective function at a point of continuity. Let this point, at least in this section, be x. Now, the value of x being regarded as fixed, let :

$$\phi(u) = \frac{f(x+u) - f(x)}{2\sin(u/2)}.$$

Here, considering the hypotheses we have chosen in this section, our intuition suggests us to work with $\phi(u)$ such that it is either continuous everywhere or at least it is continuous except for a finite number of finite jumps. By the construction of ϕ, we must analyse this function in the neighborhood of $u = 0$. We see,

$$\phi(u) = \underbrace{\frac{f(x+u) - f(x)}{u}}_{\mathcal{Q}_1} \cdot \underbrace{\frac{(1/2)u}{\sin(u/2)}}_{\mathcal{Q}_2}.$$

First of all, the quotient $\mathcal{Q}_2$, is continuous everywhere except at zero, and the limiting value of $\mathcal{Q}_2$ at zero is 1. Thus, if we define $\mathcal{Q}_2$ to be 1 at zero, the it becomes everywhere continuous, or else if we define it with any finite value other than 1 then it would have a finite jump discontinuity at zero. Either way the purpose appears to get fulfilled. Next, we observe the quotient $\mathcal{Q}_1$ near the neighbourhood of zero. To say that $\mathcal{Q}_1$ is continuous everywhere, including the point zero, it is to precisely say that $f(t)$ has a derivative, when $t = x$. On the other hand. Again, if $\mathcal{Q}_1$ is not continuous at zero but has a finite value means that the right-hand-derivative and the left-hand-derivative of $f(t)$ are both finite but are not equal to each other at $t = x$. Hence, this discussion will put some further constraints on the function f.

Continuing from (21), we get :

$$s_n^f(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}\phi(u)\sin\big[(n+1/2)u\big]du. \tag{22}$$

Now taking into account the discussions done in the last paragraph, it is clear that if, $f(t)$ is continuous everywhere except for a finite number of finite jumps in a period, then the same is true for $\phi(u)$. So we can apply (15) to (22) and get :

$$\lim_{n\to\infty}\big[s_n^f(x) - f(x)\big] = 0.$$

Thus, we may now state the following result.

Theorem 1.9.1. *Let $f(x)$, periodic with period 2π, be continuous everywhere, or be continuous except for a finite number of finite jumps in a period. If f is continuous at a point x_0 such that the left-hand-derivative and the right-hand-derivative of f at x_0 exists (whether they are same or not, is not a thing to bother about). Then, its Fourier series converges to the value $f(x_0)$.*

1.10 Uniform convergence under special hypothesis

In this section, we will consider f to be a Broken-Line Function, as introduced in section(1.5). Clearly, the conclusion deduced in section(1.9) also holds for Broken line functions. Moreover, from the discussions done in section(1.5) it is clear that the Fourier series for such a function is convergent, so it is assured that the sum of the series is $f(x)$ for all values of x. So, since the series actually represents $f(x)$, we can write the difference as :

$$f(x) - s_n^f(x) = \sum_{k=n+1}^{\infty} \Big[a_k \cos(kx) + b_k \sin(kx)\Big].$$

As a consequence of (11), we get that $\big|a_k \cos(kx) + b_k \sin(kx)\big| \leq \frac{2C}{k^2}$, and therefore :

$$\big|f(x) - s_n^f(x)\big| \leq 2C \sum_{k=n+1}^{\infty} \frac{1}{k^2}.$$

Now, whenever $k-1 \leq t \leq k$, it follows that :

$$\frac{1}{k} \leq \frac{1}{t} \leq \frac{1}{k-1} \implies \frac{1}{k^2} \leq \frac{1}{t^2}.$$

So, we can perform,

$$\begin{aligned} \frac{1}{k^2} &= \int_{k-1}^{k} \frac{dt}{k^2} \leq \int_{k-1}^{k} \frac{dt}{t^2} \\ \implies \sum_{k=n+1}^{\infty} \frac{1}{k^2} &\leq \sum_{k=n+1}^{\infty} \int_{k-1}^{k} \frac{dt}{t^2} \\ &= \int_{n}^{n+1} \frac{dt}{t^2} + \int_{n+1}^{n+2} \frac{dt}{t^2} + \cdots \\ &= \int_{n}^{\infty} \frac{dt}{t^2} \\ &= \frac{1}{n}. \end{aligned}$$

Hence, for all values of x,

$$|f(x) - s_n^f(x)| \leq \frac{2C}{n}.$$

In the inequality above, the member of the right-hand-side is independent of x, and approaches zero as n becomes infinite. Hence, by definition we conclude that, *the Fourier series for a Broken line function converges uniformly to the function for all values of x.*

1.11 Convergence at a point of discontinuity

Let f be a 2π-periodic function, such that it is continuous everywhere or is continuous except for a finite number of finite jumps in a period. Let $f(x+)$ and $f(x-)$ denote its limiting value when approached from the right and left of the point x respectively. Clearly, $f(x+)$ and $f(x-)$ might be equal or different depending on whether x is a point of continuity or discontinuity.

Now for a given x, let us construct the following functions :

$$\phi_1(u) = \begin{cases} \frac{f(x+u)-f(x+)}{2\sin(u/2)} & \text{for } u > 0, \\ 0 & \text{for } u < 0. \end{cases}$$

$$\phi_2(u) = \begin{cases} \frac{f(x+u)-f(x-)}{2\sin(u/2)} & \text{for } u < 0, \\ 0 & \text{for } u > 0. \end{cases}$$

The only thing which needs to be taken care of is the value of ϕ_1 and ϕ_2 at zero. We assume that the quotients $\frac{f(x+u)-f(x+)}{2\sin(u/2)}$ and $\frac{f(x+u)-f(x-)}{2\sin(u/2)}$ approaches a limit as u approaches 0. Basically for ϕ_1, we assume that a function which equals $f(t)$ for $t < x$ and equals $f(x+)$ for $t = x$ has a right-hand-derivative at $t = x$. Similar interpretation can be made for the other quotient.

Thus, both ϕ_1 and ϕ_2 likewise approach limits for $u = 0$, and if they are defined by their limiting values there, then they are continuous except for a finite number of finite jumps throughout $(0, \pi)$ and $(-\pi, 0)$ respectively. Further, the hypotheses on which (15) is based are fulfilled even if *phi* and *phi*^{2} are integrable over $(0, \pi)$ and is identically zero on $(-\pi, 0)$. So, under such assumption, from (15) we will have :

$$\lim_{n\to\infty} \int_0^{\pi} \phi(u) \sin\big[(n + 1/2)u\big] du = 0. \tag{23}$$

A similar discussion can be made for the case considering the interval $(-\pi, 0)$. Also, as the integral in (19) is an even function, we have :

$$\int_{-\pi}^{0} \frac{\sin\big[(n+1/2)u\big]}{2\sin(u/2)} du = \int_0^{\pi} \frac{\sin\big[(n+1/2)u\big]}{2\sin(u/2)} du = \frac{\pi}{2}.$$

Here, on multiplying by $\frac{f(x+)}{\pi}$ we get :

$$\frac{1}{2} f(x+) = \frac{1}{\pi} \int_0^{\pi} f(x+) \frac{\sin\big[(n+1/2)u\big]}{2\sin(u/2)} du,$$

and similarly,

$$\frac{1}{2} f(x-) = \frac{1}{\pi} \int_{-\pi}^{0} f(x-) \frac{\sin\big[(n+1/2)u\big]}{2\sin(u/2)} du.$$

Finally, from (18), $s_n^f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u) \frac{\sin\big[(n+1/2)u\big]}{2\sin(u/2)} du$, we get $s_n^f(x) - \frac{1}{2}[f(x+) - f(x-)] =$

$$\frac{1}{\pi} \int_{-\pi}^{0} \phi_2(u) \frac{\sin\big[(n+1/2)u\big]}{2\sin(u/2)} du + \frac{1}{\pi} \int_0^{\pi} \phi_1(u) \frac{\sin\big[(n+1/2)u\big]}{2\sin(u/2)} du.$$

Hence, using (23) here we get :

$$\lim_{n\to\infty} s_n^f(x) = \frac{1}{2}[f(x+) + f(x-)].$$

Thus, we can conclude this section by presenting the following result.

Theorem 1.11.1. *For any periodic function f with period 2π, which is either continuous or continuous everywhere except for a finite number of finite jumps, its Fourier series at x (it may or may not be a point of continuity) converges to the value $\frac{1}{2}[f(x+) + f(x-)]$ such that the right-hand-derivative and the left-hand-derivative of f exists at x.*

Note that the above conclusion also holds true for f being piece-wise continuous. Direct proof for this can be found at [10].

1.12 A theoretical & numerical case study

Let us analyse the 2π periodic function f which is identically equal to the identity function in the interval $(-\pi, \pi)$. The graph of this function is given in figure(2) below.

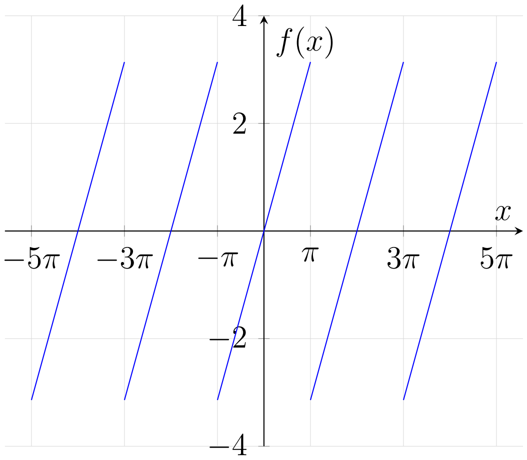


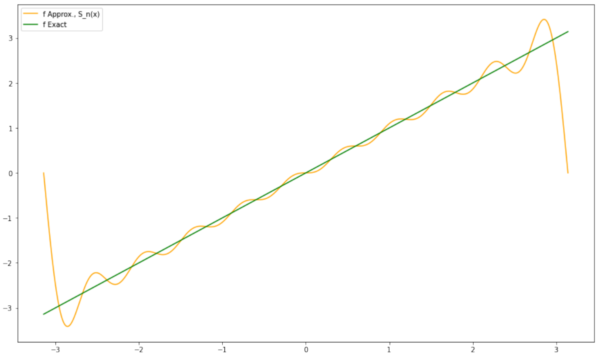
Figure 2: Graph of the 2π-periodic function, such that $f(x) \equiv x$ for $x \in (-\pi, \pi)$

Since, $f \equiv x$ in the interval $(-\pi, \pi)$, which is an odd function, so by using the conclusions made in section(1.4), we know that the Fourier coefficients a_k for all k are zero. Again, using integration-by-parts :

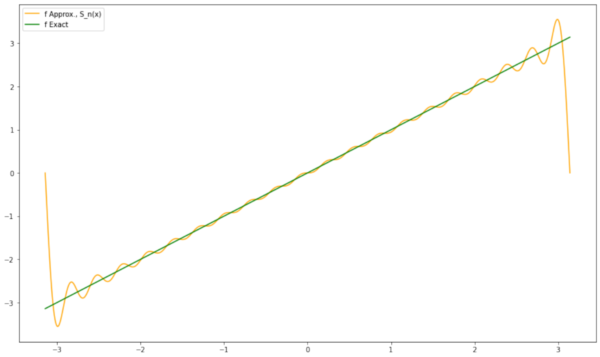
$$\int_0^\pi x\sin(kx)dx = [-\frac{1}{k}x\cos(kx)]\Big|_0^\pi + \frac{1}{k}\int_0^\pi \cos(kx)dx = \frac{-\pi}{k}\cos(k\pi)$$
$$\implies b_k = \frac{2}{\pi}\int_0^\pi x\sin(kx)dx = (-1)^{k-1}\frac{2}{k}.$$

Hence, the Fourier series has the form :

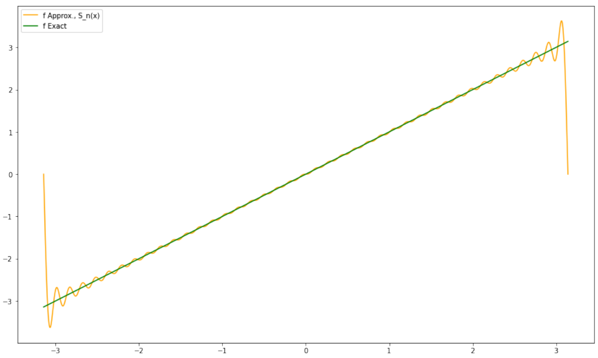
$$2\big[\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4} + \cdots\big] \tag{24}$$



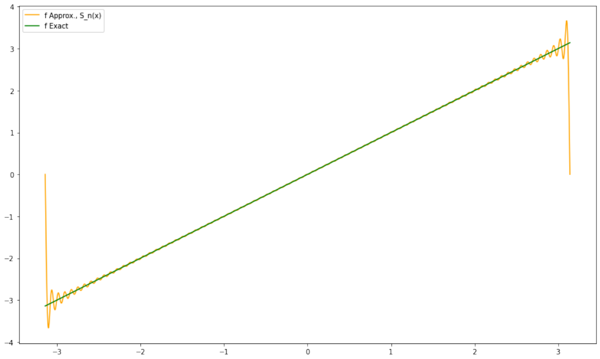
(a) Plot for $s_{10}^f(x)$ and $f(x)$



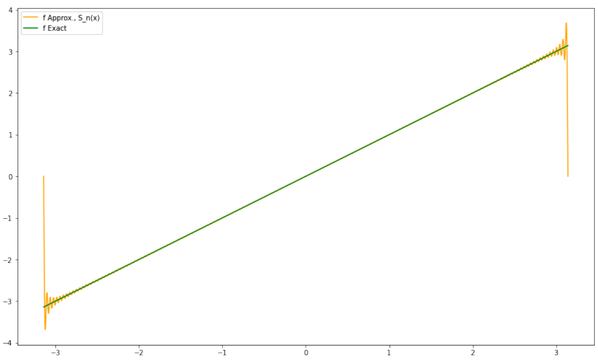
(b) Plot for $s_{20}^f(x)$ and $f(x)$



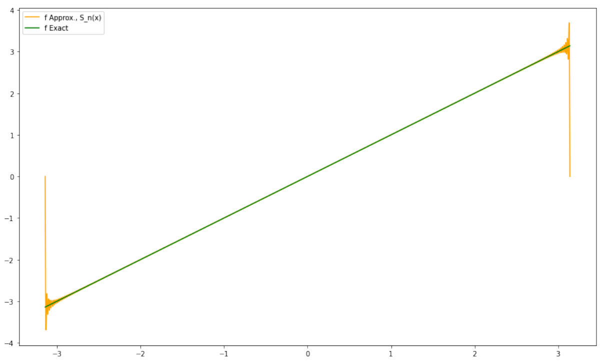
(c) Plot for $s_{40}^f(x)$ and $f(x)$



(d) Plot for $s_{80}^f(x)$ and $f(x)$



(e) Plot for $s_{160}^f(x)$ and $f(x)$



(f) Plot for $s_{320}^f(x)$ and $f(x)$

Figure 3: Plots of $s_n^f(x)$ and $f(x)$ for different truncation terms, n with 5000 equidistant evaluation points in $[-\pi, \pi]$.

Now, in (24) each term is odd, still having the same period 2π, hence its sum must also be odd and 2π-periodic, which here is in accordance with the nature of the function f. The convergence of this series (24) to the function f verifies the result achieved in the end of Section(1.9) and Section(1.10).

Further, the graph clearly tells that f is discontinuous at $x = \pm\pi, \pm 3\pi, \cdots$. One can see that, at these points of discontinuity, the series (24) converges to zero as each term vanishes individually. Thus, the result produced at the end of Section(1.11) is also verified, which says that the sum of (24) must be zero at all these points of discontinuity.

The results which have been verified through this example in the last two paragraphs, can also be verified numerically‡ . The convergence of the Fourier series to the function is numerically verified here, as we can see in figure(3e)[truncation term, n=160] and figure(3f)[truncation term, n=320] that, we cannot even differentiate the plots of $s^f_{160}(x)$ and $f(x)$; and $s^f_{320}(x)$ and $f(x)$ respectively. Further the result achieved for a point of discontinuity, at the end of section(1.11), can also be verified, as in each plot the curve for $s^f_n(x)$ passes through $y = 0$ for the endpoints $x = \pm\pi$.

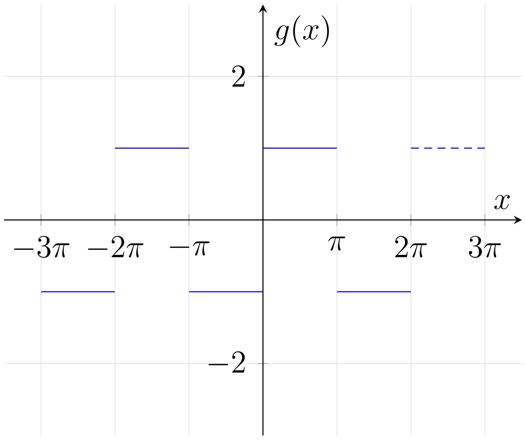
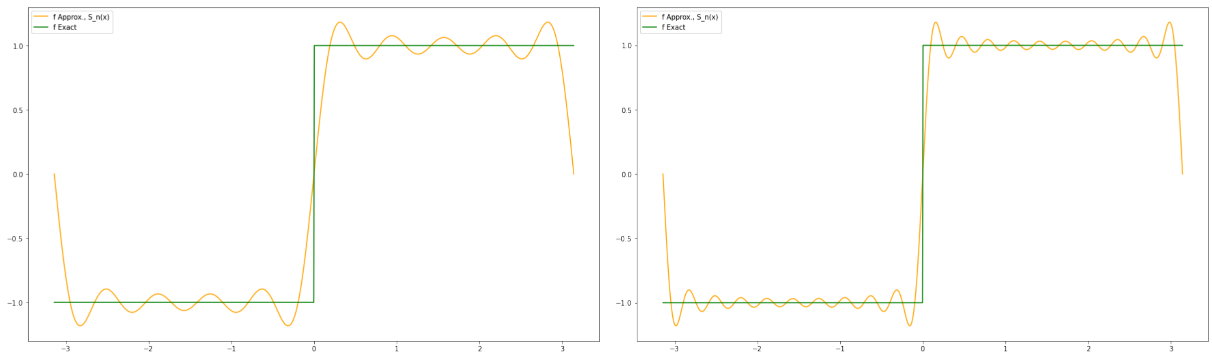


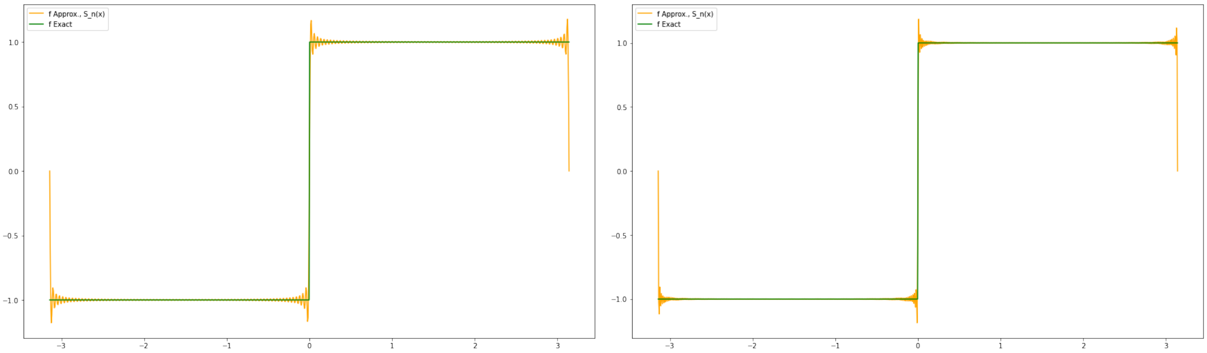
Figure 4: Graph of the 2π-periodic extension of the Signum function in $(-\pi, \pi)$

Next, let us consider $g(x)$ to be the function which is the 2π-periodic extension of the *Signum function* in the interval $(-\pi, \pi)$. The graph of this function is given in Figure(4) above. Here, clearly the points $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \cdots$ are points of discontinuities, and the result at the end of Section(1.11) says that $s^g_n(x)$ at these points must be zero. Further the results deduced for convergence at points of continuity says that $s^g_n(x)$ must converge to $g(x)$ at points other than these mentioned points of discontinuities. Numerical verification of these results is evident from plots at Figure(5) and Figure(6).



(a) Plot for $s^g_{10}(x)$ and $g(x)$ (b) Plot for $s^g_{20}(x)$ and $g(x)$

Figure 5: Plots of $s^g_n(x)$ and $g(x)$ for different truncation terms, n with 1000 equidistant evaluation points in $[-\pi, \pi]$.



(a) Plot for $s^g_{160}(x)$ and $g(x)$ (b) Plot for $s^g_{320}(x)$ and $g(x)$

Figure 6: Plots of $s^g_n(x)$ and $g(x)$ for different truncation terms, n with 1000 equidistant evaluation points in $[-\pi, \pi]$.

1.13 Sufficiency of conditions relating to a restricted neighbourhood

We continue our discussion from Section(1.9) in order to deduce a conclusion of higher degree of generality from (22). Let us consider a 2π-periodic function f such that $f(t)$ and $[f(t)]^2$ are both integrable over a period, and further there is an interval $(x-h, x+h)$ throughout which $f(t)$ vanishes identically. This clearly tells both $\phi(u)$ and $[\phi(u)]^2$ are integrable from $-\pi$ to π, since the numerator of $\phi(u)$ is identically zero throughout the neighborhood of the point where the denominator vanishes. So, in that interval :

$$\lim_{n\to\infty} s^f_n(x) = f(x) = 0.$$

Now let $f_1(t)$ and $f_2(t)$ be two functions, each along with their squares integrable over a period, and let these be identically equal to each other throughout that interval $(x-h, x+h)$. If $s^{f_1}_n(x)$ and $s^{f_2}_n(x)$ are their Fourier partial sums respectively at x, then clearly $s^{f_1}_n(x) - s^{f_2}_n(x)$ is the partial sum of the Fourier series for $(f_1 - f_2)(x)$. In that certain interval $(x-h, x+h)$, discussed till now in this section, $(f_1 - f_2)(x)$ is identically zero. Thus, by the discussion of the first paragraph of this section, we have :

$$\lim_{n\to\infty} s^{f_1}_n(x) - s^{f_2}_n(x) = (f_1 - f_2)(x) = 0.$$

Thus, in that neighbourhood, if one partial sum converges to a certain limit, then the other partial sum converges to the same limit. Hence, this implies the conclusion that, *the convergence of the Fourier series for a function at a specified point depends only on the behaviour of the function in the neighbourhood of the point.*

Basically, if any square integrable function f is given then in order to find the sum of its Fourier series in a given point x, one would just need another known function g such that in a small neighborhood of x, functions f and g are identical. The sum of the Fourier series of g would give the required result.

1.14 Some more results

In this section we state some significant results related to the Fourier series of a function without going through their detailed proofs. Let us proceed ahead by stating the following definitions.

Definition 1.14.1. *We define $Si(x)$, the Sine Integral as :*

$$Si(y) = \int_0^y \frac{\sin(x)}{x} dx, \qquad \text{for } y \in \mathbb{R}_{>0}.$$

1.14.1 Gibbs Phenomenon

In 1899 Gibbs[16] pointed out that the *approximation curves* or the Fourier partial sum for the series (24), behaves in an unique way at the points of discontinuity $\pm\pi$. He stated, in effect, that the curve of $s_n^f(x)$, for large values of n, falls from the point $(-\pi, 0)$ at a steep gradient to a point very nearly at a depth of $2Si(\pi)$ below the X-axis, and then oscillates above and below the curve of $f(x)$, close to this curve until x approaches π, when it falls from a point very nearly at a height $2Si(\pi)$ above the X-axis at a steep gradient to $(\pi, 0)$. Interestingly, his statement was not accompanied by any proof.

In 1906, Bôcher in a memoir on Fourier's Series[3], greatly extended Gibbs's result. He showed, among other things, that the phenomenon which Gibbs had observed in the case of this particular Fourier's Series holds in general at ordinary points of discontinuity. Quoting the result, we present the following proposition.

Proposition 1.14.2. *If $f(x)$ is 2π-periodic and in a finite interval has no discontinuities other than a finite number of finite jumps, and if it has a derivative which in any finite interval has no discontinuities other than a finite number of finite discontinuities, then as n becomes infinite the approximation curve $y = S_n^f(x)$ approaches uniformly the continuous curve made up of :*

1. *the discontinuous curve $y = f(x)$,*
2. *an infinite number of straight lines of finite lengths parallel to the Y-axis and passing through the points $a_1, a_2, \cdots$ on the X-axis where the discontinuities of $f(x)$ occur. If a is any one of these points, the line in question extends between the two points whose ordinates are given by :*

$$f(a-) + \frac{\mathcal{J} Si_0}{\pi}, \qquad f(a+) - \frac{\mathcal{J} Si_0}{\pi},$$

where $\mathcal{J}$ is the jump if $f(x)$ at a, i.e, $\mathcal{J} = f(a+) - f(a-)$ and $Si_0 = \int_\pi^\infty \frac{\sin(x)}{x} dx \approx$ -0.2811.

This overshooting and undershooting behaviour of $s_n^f(x)$ near a point of jump discontinuity can can be observed in the plots given in figure(3), figure(5) and figure(6).

Continuing this discussion along with using the conclusions of section(1.11), one can present a much formal statement of Gibbs phenomenon. Avoiding a detailed proof for the same, we present the following theorem.

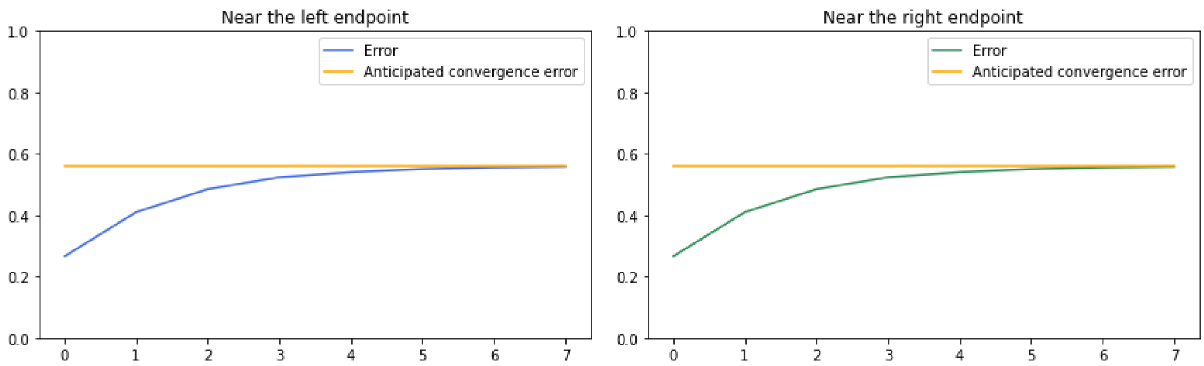
Theorem 1.14.3. ***Gibbs Phenomenon*** *For a 2π-periodic function f, which is piece-wise continuous, such that x_0 is a point of jump discontinuity, we have :*

$$\lim_{n\to\infty} s_n^f(x_0+\frac{\pi}{n}) - f(x_0+) = (\frac{1}{\pi}Si(\pi)-\frac{1}{2})\big|f(x_0+)-f(x_0-)\big|$$
$$\lim_{n\to\infty} s_n^f(x_0-\frac{\pi}{n}) - f(x_0-) = (\frac{1}{\pi}Si(\pi)-\frac{1}{2})\big|f(x_0+)-f(x_0-)\big|.$$

Basically theorem(1.14.3) implies that, *the error in approximation of $f(x)$, by the approximating function or the Fourier partial sum $s_n^f(x)$ becomes constant near the points of jump discontinuities. The constant error of approximation is* 0.9% *of the jump present at the point.* It is because in the right-hand-side of the equations, we have :

$$\underbrace{(\frac{1}{\pi}Si(\pi)-\frac{1}{2})}_{\approx 0.0894,\text{ the Wilbraham constant}} \quad \underbrace{\big|f(x_0+)-f(x_0-)\big|}_{\text{jump at } x_0}.$$

For numerical experiment, we consider f as introduced in the first paragraph of section(1.12). We approximate $f(x)$ by the truncated Fourier sum $s_n^f(x)$, starting with $n = 10, 20, 40, 80, \cdots, 1280$. The plots in figure(7) shows that the error in approximation of $f(x)$ by $s_n^f(x)$, at the left-point of jump discontinuity $x = -\pi$ and the right-point of jump discontinuity $x = \pi$, approaches the value 0.9% of the jump as the truncation number n becomes larger.



(a) Plot for error in approximation near $-\pi$ and the anticipated error in approximation,i.e., 0.9% of jump at $-\pi$.

(b) Plot for error in approximation near π and the anticipated error in approximation,i.e., 0.9% of jump at π.

Figure 7: Plots for error in approximation near $-\pi$ and π as truncation term n becomes larger and the anticipated error in approximation,i.e., 0.9% of jump at $-\pi$ and π.

1.14.2 Weierstrass theorem for trigonometric approximation

Definition 1.14.4. *An expression of the form :*

$$\frac{\alpha_0}{2} + \alpha_1\cos(x) + \alpha_2\cos(2x) + \cdots + \alpha_n\cos(nx)$$
$$+ \beta_1\sin(x) + \beta_2\sin(2x) + \cdots + \beta_n\sin(nx),$$

is called an n^{th} order ***trigonometric sum****, if α_n and β_n are not both zero.*

Analogous to the Weierstrass Approximation Theorem presented in [1], we have the following theorem.

Theorem 1.14.5. *Any 2π-periodic continuous function f can be uniformly approximated by a trigonometric sum, with any preassigned degree of accuracy. Basically, for any given $\epsilon > 0$, for any 2π-periodic continuous function f, $\exists$ a trigonometric sum $T(x)$ of some order, such that :*

$$\big|f(x) - T(x)\big| < \epsilon \qquad \forall x. \tag{25}$$

It must be noted that if the Fourier series for $f(x)$ is uniformly convergent, then (25) is satisfied by taking the Fourier partial sum $s_n^f(x)$ for $T(x)$. But there exists continuous functions whose Fourier series diverges, and hence the existence of such functions gives significance to theorem(1.14.5).

Also, continuing from (25), since the result holds $\forall x \in X$ we get that :

$$\underset{X}{sup}\big|f(x) - T(x)\big| < \epsilon.$$

Further, it can be observed that any power of sine or cosine can be written as a trigonometric sum. Hence we can conclude that, *the space of all trigonometric polynomials is dense in the space of all 2π-periodic continuous function.*

1.14.3 Least square property & Parseval's Identity

Let f be a function such that it is square integrable in $(-\pi, \pi)$. The **least square property** says that, *when considered as an approximation to $f(x)$, the partial sum of its Fourier series is distinguished among all trigonometric sums of n^{th} order at most, as the one for which the integral of the square of the error in approximation is minimum.* Basically, for any $\mathrm{n^{th}}$ order trigonometric sum and mentioned function f,

$$\int_{-\pi}^{\pi} \big|f(x) - s_n^f(x)\big|^2 dx \le \int_{-\pi}^{\pi} \big|f(x) - t_n(x)\big|^2 dx \tag{26}$$

We know that, for a proper closed subspace $M \subset X$ of a normed space X, if $x \in X \setminus M$, then $dist(x, M) > 0$. Here, if d is the metric induced by the norm on X, then $dist(x, M) := \inf_{y \in M} d(x, y)$. In this context, when we have the following :

$X = L^2[-\pi, \pi]$,
$M = span\{1, \cos(x), \cos(2x), \cdots, \cos(nx), \sin(x), \sin(2x), \cdots, \sin(nx) |$ for some $n \in \mathbb{N}\}$,
$f \in X \setminus M$,

then the least square property tells that, we have $dist(f, M) = \|f(x) - s_n^f(x)\|_2$.

Further, for 2π-periodic continuous functions, where a_k and b_k are its Fourier coefficients, then we have the *Parseval's Identity* :

$$\boxed{\frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 \; dx.}$$

Checkpoint. Now that we have seen the Parseval's Identity, we can answer a question which has been lurking in the background since Section(1.13). The indicated question is *what can be commented about two given different functions from appropriate function classes, which have all of their Fourier coefficients exactly the same?* If we have such two functions, say f and g, then by Paseval's Identity we get that $||f||_{L^2} = ||g||_{L^2}$. Hence, any two functions from appropriate function classes, if at all has each of their Fourier cefficients to be same, then they can differ at most by a measure zero set.

CHAPTER 2: STURM-LIOUVILLE SYSTEMS - I

2.1 Introduction

The purpose of this section is to mainly to settle the prelude for looking out for *Sturm-Liouville systems.* To begin with, we know that *Linear differential operators* are linear operators which have a differential form, that is, it typically involves derivatives of the function(s) it operates on. For example, the most common linear differential operator one can thing about is the first-order derivative operator, denoted as d/dx, which operates on a function $f(x)$ to produce its derivative $f'(x)$. The second-order derivative operator (d^2/dx^2) and the Laplace Operator (∇^2 $or \Delta$) are also some of the most familiar linear differential operators, one often deals with. Firstly, we start the discussion with self-adjoint differential operators (DO) and then we will get familiar with DO associated with SL differential equations, which fall under the hood of self-adjoint DOs.

2.1.1 Exact second-order DE & Integrating Factor

Definition 2.1.1. ***Exact second-order differential equation***
The second-order homogeneous linear differential equation :

$$L[u] = p_0(x)u''(x) + p_1(x)u'(x) + p_2(x)u(x) = 0 \tag{27}$$

is said to be ***exact*** *if and only if, for some* $A(x), B(x) \in C^1$*, we have :*

$$p_0(x)u''(x) + p_1(x)u'(x) + p_2(x)u(x) = \frac{d}{dx}[A(x)u' + B(x)u] \tag{28}$$

for all functions $u \in C^2$*.*

Definition 2.1.2. ***Integrating Factor*** *An integrating factor for the DE(27) is a function* $v(x)$ *such that* $vL[u]$ *is exact.*

Note that, here onward it is assumed that $p_0 \in C^2$, $p_1 \in C^1$ and $p_2 \in C$. If, an integrating factor v for (27) can be found, then clearly we have,

$$v(x)[p_0(x)u''(x) + p_1(x)u'(x) + p_2(x)u(x)] = \frac{d}{dx}[A(x)u' + B(x)u].$$

Hence the solutions of the second-order homogeneous linear differential equation (27) are also the solutions of the first-order inhomogeneous linear differential equation

$$A(x)u' + B(x)u = C, \tag{29}$$

where C is an arbitrary constant. Also, the solution of the inhomogeneous differential equation $L[u] = r(x)$ are those of the first-order differential equation :

$$A(x)u' + B(x)u = \int v(x)r(x)dx + C. \tag{30}$$

Basically, we now have that second-order differential equations like (27) and $L[u] = r(x)$ can be solved simply by looking for the solutions of first-order differential equations like (29) and (30). Hence, this reduction in complexity of finding solutions provides meaning to the act of looking for exact second-order linear differential equation in the first place.

Now continuing from the discussions ahead, we have :

$$\frac{d}{dx}[A(x)u' + B(x)u] = A(x)u'' + (A'(x) + B(x))u' + B'(x)u.$$

Thus, by simply comparing the coefficients of u and its derivatives we know, DE(differential equation) (27) is exact if and only if $p_0 = A$, $p_1 = A' + B$, $p_2 = B'$. Hence, DE(27) is exact iff :

$$\begin{aligned} p_2 = B' = (p_1 - A')' = p_1' - A'' = p_1' - p_0'' \\ \iff p_0'' = p_1' + p_2 = 0. \end{aligned}$$

Thus, by using the discussion above we can present the following lemma and further, as a direct consequence of it we can also state a corollary as follows.

Lemma 2.1.3. *The DE(27) is exact iff its coefficient functions satisfy:*

$$p_0'' = p_1' + p_2 = 0.$$

Corollary 2.1.4. *A function $v \in C^2$ is an integrating factor for the DE(27) if and only if it is a solution of the second-order homogeneous linear DE :*

$$M[v] = [p_0(x)v]'' - [p_1(x)v]' + p_2(x)v = 0. \tag{31}$$

2.1.2 Adjoint of a DE & Lagrange Identity

We proceed ahead by presenting the following definition.

Definition 2.1.5. ***Adjoint*** *The operator M in (31) is called the adjoint of the linear operator L. The DE(31), expanded to the DE :*

$$p_0v'' + (2p_0' - p_1)v' + (p_0'' - p_1' + p_2)v = 0, \tag{32}$$

is called the adjoint of the DE(27).

Now, let us consider (32), as $q_0v'' + q_1v' + q_2v = 0$, such that :

$$\begin{aligned} q_0 &= p_0 \\ q_1 &= 2p_0' - p_1' \\ q_2 &= p_0'' - p_1' + p_2, \end{aligned}$$

and if we compute the adjoint then we get :

$$\begin{aligned} q_0w'' + (2q_0' - q_1)w' + (q_0'' - q_1' + q_2)w = 0 \\ p_0w'' + p_1w' + p_2w = 0. \end{aligned}$$

Therefore, this result constitutes to the following lemma.

Lemma 2.1.6. *Adjoint of the adjoint of a given second-order linear differential equation is again the original given differential equation.*

As we under the assumption that $p_0 \in C^2$ and $p_1 \in C^1$, so we have :

$$\begin{aligned} vL[u] - uM[v] &= (vp_0)u'' - u(p_0 v)'' + (vp_1)u' + u(p_1 v)' \\ &= \frac{d}{dx}[p_0(u'v - uv') - (p_0' - p_1)uv]. \end{aligned} \tag{33}$$

Here, we have (7) as the **Lagrange Identity**, that is :

$$vL[u] - uM[v] = \frac{d}{dx}[p_0(u'v - uv') - (p_0' - p_1)uv]. \tag{34}$$

One must note that the left-side of (34) is thus always an exact differential of a homogeneous bilinear expression in u, v, and their derivatives.

2.1.3 Self-Adjoint DE

Definition 2.1.7. ***Self-Adjoint DE*** *Homogeneous linear differential equations that coincide with their adjoint are called Self Adjoint.*

Naturally from (32), we have $2p_0' - p_1 = p_1 \implies p_0' = p_1$. Since this clearly implies that $p_0'' - p_1' = 0 \implies p_0'' - p_1' + p_2 = p_2$, we can conclude that for a given DE(27) to be self-adjoint we have the following :

$$\begin{aligned} \text{Necessary condition : } & p_0' = p_1 \\ \text{Sufficient condition : } & p_0'' = p_1'. \end{aligned}$$

For self-adjoint case, one must observe that the last term in the Lagrange Identity (34) vanishes. Moreover, when DE(27) is reduced to the normal form as $u''(x) + \frac{p_1}{p_0}u'(x) + \frac{p_2}{p_0}u(x) = 0$, we see that the DE $hu'' + (ph)u' + (qh)u = 0$ is self adjoint if and only if $h' = ph \implies h = e^{\int p dx}$. This discussion leads us to present the following theorem, which characterises the form of self-adjoint differential equations, and also gives a method to make a DE(27) self-adjoint.

Theorem 2.1.8. *The second order linear differential equation (27) is self-adjoint if and only if it has the form :*

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = 0 \tag{35}$$

Further, a DE(27) can be made self-adjoint by multiplying it through by :

$$h(x) = \left[e^{\int \frac{p_1}{p_0} dx}\right] / p_0.$$

For self-adjoint differential equations(35), the Lagrange Identity simplifies to :

$$vL[u] - uM[v] = \frac{d}{dx}[p(x)(u'v - uv')]. \tag{36}$$

Example. We consider the Chebyshev DE as follows :

$$(1-x^2)u'' - xu' + \lambda u = 0. \tag{37}$$

After comparing this with the standard form of second-order linear homogeneous differential equation as (27) we get the following :

$$\begin{aligned} p_0 &= 1-x^2 \\ p_1 &= -x \\ p_2 &= \lambda. \end{aligned}$$

Therefore, we now compute the function $h(x)$ as given in Theorem(2.1.8) as,

$$\begin{aligned} h(x) &= e^{\int \frac{p_1}{p_0}dx}/p_0 \\ &= e^{\int \frac{-x}{1-x^2}dx}/(1-x^2) \\ &= e^{\log(\sqrt{1-x^2})+C}/(1-x^2) \qquad\qquad [\text{where } C \text{ is a constant}]. \\ &= \frac{e^C(\sqrt{1-x^2})}{(1-x^2)} \\ &= \frac{e^C}{\sqrt{1-x^2}}. \end{aligned}$$

Thus, the self-adjoint form is :

$$\begin{aligned} &(e^C\frac{1-x^2}{\sqrt{1-x^2}})u'' + (e^C\frac{(-x)}{\sqrt{1-x^2}})u' + (e^C\frac{\lambda}{\sqrt{1-x^2}})u = 0 \\ \implies & e^C\frac{d}{dx}\Big[\sqrt{1-x^2}\frac{du}{dx}\Big] + e^C\Big[\frac{\lambda}{\sqrt{1-x^2}}\Big]u = 0. \end{aligned}$$

Thus the self-adjoint form of the Chebyshev Differential Equation (37) is :

$$\boxed{\frac{d}{dx}\Big[\sqrt{1-x^2}\cdot\frac{du}{dx}\Big] + \Big[\frac{\lambda}{\sqrt{1-x^2}}\Big]u = 0.}$$

2.1.4 Sturm-Liouville Systems

A Sturm-Liouville equation is a second-order homogeneous linear differential equation of the form :

$$\frac{d}{dx}\big[p(x)\frac{du}{dx}\big] + [\lambda\rho(x) - q(x)]u = 0. \tag{38}$$

In operational notation, with $L = D[p(x)D] - q(x)$, we can write (12) in the following compact form :

$$L[u] + \lambda\rho(x)u = 0.$$

We see that DEs like (38) are self-adjoint for real λ; further to ensure the existence of solutions, the functions q and ρ are assumed to be continuous and $p \in C^1$. For a given value of λ, (38) defines a linear operator transforming any function $u \in C^2$ into $L[u]+\lambda\rho u$. The Sturm-Liouville equation is called *Regular* in a closed finite interval $a \le x \le b$ when the functions $p(x)$ and $\rho(x)$ are positive for $a \le x \le b$.

Definition 2.1.9. ***Sturm-Liouville System*** *A Sturm-Liouville system (or SL system) is a Sturm-Liouville equation together with endpoint conditions (or boundary conditions), to be satisfied by the solutions, for example $u(a) = u(b) = 0$. Further, a regular SL system is a regular SL equation (38) on a finite closed interval $[a, b]$, together with two separated endpoint conditions of the form :*

$$\alpha u(a) + \alpha' u'(a) = 0; \qquad \beta u(b) + \beta' u'(b) = 0. \tag{39}$$

Here $\alpha, \alpha', \beta, \beta'$ are given real numbers. We obviously exclude the two trivial conditions $\alpha = \alpha' = 0$ and $\beta = \beta' = 0$.

A non-trivial solution of an *SL system* is called an ***eigenfunction***, and the corresponding λ is called ***eigenvalue***. The set of all eigenvalues of a regular SL system is called the ***spectrum*** of the system.

Example Let us consider the SL system consisting of the DE $u'' + \lambda u = 0$, in the interval $[0, \pi]$ with *Dirichlet boundary conditions* $u(0) = u(\pi) = 0$.

Here, firstly we consider the **case :** $\lambda < 0$. On solving the characteristic equation we get the roots r as $r = -\lambda = \alpha^2$, where $\alpha \in \mathbb{R}_{>0}$. Hence, the general solution for the above DE would be $u(x) = Ae^{\alpha x} + Be^{-\alpha x}$. Imposing the boundary conditions we get $A = B = 0$, hence the only solution for the given SL system under this case is the trivial solution $u \equiv 0$. Next, Here, we consider the **case :** $\lambda = 0$. On solving the characteristic equation we get the roots r as $r = 0$ and hence the general solution would be $u(x) = A + Bx$. Again on imposing the boundary conditions we get the only solution for this case to be the trivial solution $u \equiv 0$. Thus, we may conclude that for the given SL system, its *eigenvalues* has to be positive.

Finally, we consider the **case :** $\lambda > 0$. Here, the roots of the characteristic equation are $r = \pm i\sqrt{\lambda}$. Thus, the general solution would be $u(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$. Here after imposing the boundary conditions we get the eigenvalues to be $\lambda = n^2, n = 1, 2, 3, \cdots$ and the eigenfunctions $u_n(x) = \sin(\sqrt{\lambda}x) = \sin(nx)$.

Definition 2.1.10. ***Periodic Endpoint Conditions*** *For SL equations whose coefficients are periodic functions of x with period $b - a$, periodic endpoint conditions :*

$$u(a) = u(b), \qquad u'(a) = u'(b), \tag{40}$$

are sometimes imposed, and this gives another SL system, called a periodic SL system.

2.2 Sturm-Liouville Series - an example

Example. Let us consider the SL system consisting of the DE $u'' + \lambda u = 0$, in the interval $[-\pi, \pi]$ with the *periodic endpoint conditions* $u(-\pi) = u(\pi)$, and $u'(-\pi) = u'(\pi)$.

Here solving this similar to the previous example we get the eigenfunctions to be $1, \cos(nx)$ and $\sin(nx)$ where n is any positive integer. Also, the corresponding eigenvalues are $\lambda = n^2$, the squares of integers.

An important observation here worth noting is that, if $n > 0$, then we have two linearly independent eigenfunctions having the same eigenvalue n^2. Also, these eigenfunctions are orthogonal using the orthogonality definition(1.2.1) introduced in section (1.2). Also, the eigenfunctions in this example are precisely the functions used in the theory of Fourier Series, Chapter (1).

Definition 2.2.1. *Two integrable real-valued functions f and g are orthogonal with weight function $\rho > 0$ on a finite interval I if and only if :*

$$\int_I \rho(x)f(x)g(x)dx = 0.$$

Let us assume that u and v satisfy an SL - equation (38) on a closed interval $a \leq x \leq b$, for values λ and μ of the parameter. For such u and v, we consider the Lagrange's Identity (33) :

$$\begin{aligned} uL[v] - vL[u] &= \frac{d}{dx}\big[p(x)[u(x)v'(x) - u'(x)v(x)]\big] \\ \implies \int_a^b uL[v] - vL[u]dx &= \int_a^b \frac{d}{dx}\big[p(x)[u(x)v'(x) - u'(x)v(x)]\big]dx \\ &= \big[p(x)[u(x)v'(x) - u'(x)v(x)]\big]\Big|_{x=a}^{x=b} \end{aligned}$$

Also,

$$\begin{aligned} \int_a^b uL[v] - vL[u]dx &= \int_a^b \big[u(-\mu\rho v) - v(-\lambda\rho u)\big]dx \\ &= \int_a^b \big[-\mu\rho uv + \lambda\rho uv\big]dx \\ &= \int_a^b \big[(\lambda - \mu)\rho uv\big]dx \\ &= (\lambda - \mu)\int_a^b \rho(x)u(x)v(x)dx \end{aligned}$$

Thus, we get :

$$(\lambda - \mu)\int_a^b \rho(x)u(x)v(x)dx = \underbrace{\big[p(x)[u(x)v'(x) - u'(x)v(x)]\big]\Big|_{x=a}^{x=b}}_{\text{we call this term its } \textbf{Boundary Term.}}. \tag{41}$$

We can present this discussions in the form of the following lemma.

Lemma 2.2.2. *If u and v satisfy an SL equation (38) on a closed finite interval $a \leq x \leq b$ for values λ and μ of the parameter, then*

$$(\lambda - \mu)\int_a^b \rho(x)u(x)v(x)dx = \big[p(x)[u(x)v'(x) - u'(x)v(x)]\big]\Big|_{x=a}^{x=b}.$$

Now continuing the discussion, after imposing the separated endpoint conditions ($\alpha \neq \alpha' \neq 0$) discussed above, we get :

$$\begin{aligned}
&\alpha u(a) = -\alpha' u'(a) \\
&\alpha v(a) = -\alpha' v'(a) \\
\implies &(-\alpha\alpha')u(a)v'(a) = (-\alpha\alpha')v(a)u'(a) \\
\implies &u(a)v'(a) - v(a)u'(a) = 0 \\
\implies &p(a)[u(a)v'(a) - u'(a)v(a)] = 0.
\end{aligned}$$

Similarly, the formulas cover the right-hand member of (41) at $x = b$, that is, $p(b)[u(b)v'(b) - u'(b)v(b)] = 0$. Therefore, we conclude that for SL system (39), the right-hand side member of (41) vanishes. Basically, for $\lambda \neq \mu$,

$$\underbrace{(\lambda - \mu)}_{non-zero} \int_a^b \rho(x)u(x)v(x)dx = 0.$$

Observing both, the last example in the previous section and the example above, we can highlight the importance of the boundary condition involved in a SL system. In these examples a change in the boundary condition itself changed the eigenfunctions. Further, lemma(2.2.2) tells that its right-hand side member, defined as the **boundary term** in (41), goes to zero only for certain and not all boundary conditions. *Basically, we can suggest that the orthogonality of the eigenfunctions of a SL system depends on its boundary condition.* Here, we have seen that for a separated endpoint condition the *boundary term* vanishes. Also, using the discussion so far, we can present the following theorem.

Theorem 2.2.3. *Eigenfunctions of a regular SL system (39) with separated boundary condition, having different eigenvalues are orthogonal with weight functions ρ. Basically, if u and v are eigenfunctions belonging to distinct eigenvalues λ and μ, then*

$$\int_a^b \rho(x)u(x)v(x)dx = 0.$$

Since periodic endpoint condition is a separated boundary condition, we have the following corollary.

Corollary 2.2.4. *The result of the theorem above also holds for SL systems with periodic endpoint conditions.*

It is easy to see that the remark made after the example discussed in this section is a proof of the fact that the converse of theorem(2.2.3) is not true.

Naturally, now it is a food for thought whether the orthogonality relation just proved above enables us to obtain expansions similar to Fourier series for general functions $f(x) \in L^2[a, b]$, but now consisting eigenfunctions of other SL systems instead of just sinusoids; resulting an infinite series called *Sturm-Liouville Series.*

2.3 Singular Sturm-Liouville Systems

Having theorem(2.2.3) for regular SL systems it is now natural to find out a similar result for SL system which are not regular. Let us now shift to a higher perspective and consider SL systems which are not regular. An SL equation (38) can be given on a finite, semi-infinite or an infinite interval I. In the finite case, I may exclude either one or both endpoints. Let us consider the following scenarios regarding an SL systems.

1. If I is semi-infinite or infinite.
2. If I is finite but both p and ρ, or one of them vanishes at one or both endpoints.
3. If q is discontinuous in one or both endpoints.
4. If $\lim_{x\to a} p(x) = 0$; $\lim_{x\to a} \rho(x) = 0$ at endpoint a.

In all these above mentioned cases, one cannot obtain a regular SL system from DE(38). In such cases, the given SL equation of the form (38) is called *Singular*, and we can obtain a *Singular SL system* from it by imposing such endpoint conditions which cannot always be described by formulas of the form (39). A common example is, the condition that a solution u be bounded near a singular endpoint.

Example. We consider the SL system consisting of the Legendre DE $[(1-x^2)u']'+\lambda u = 0$, in the interval $-1 < x < 1$ together with the endpoint condition that a solution u be bounded near the endpoints. This forms a singular SL system.

In order to solve this system, we employ the *power series method*[22]. Firstly, we identify $x = 0$ as a *regular point*[22] and then solve the DE. For non-negative integers n, the eigenvalues are $\lambda_n = n(n+1)$ and the general solution of the Legendre DE becomes,

$$y(x) = a_0\phi_\lambda(x) + a_1\psi_\lambda(x), \qquad \text{for some constants } a_0, a_1 \text{ such that,}$$

$$\phi_\lambda(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{2k}}{(2k)!} \prod_{m=1}^{k} \{(2m-1)(2m-2)-\lambda\},$$

$$\psi_\lambda(x) = x + \sum_{k=1}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \prod_{m=1}^{k} \{(2m)(2m-1)-\lambda\}.$$

It can be verified that $\phi_\lambda(x)$ and $\psi_\lambda(x)$ are linearly independant as their Wronskian at $x = 0$ is 1. From the above forms of ϕ_λ, ψ_λ and now that λ depends on n, it is evident that when n is even the series $\phi_n(x)$ terminates with x^n, whereas the series for $\psi_n(x)$ does not terminate. When n is odd, it is the series for $\psi_n(x)$ which terminates with x^n, while that for $\phi_n(x)$ does not terminate. In the first case when n is even, $\phi_n(x)$ is a polynomial of degree n. The same is true for $\psi_n(x)$ in the second case when n is odd. Therefore, for any nonnegative integer n, either $\phi_\lambda(x)$ or $\psi_\lambda(x)$, but not both, is a polynomial of degree n. It follows that the general solution of the Legendre DE contains a polynomial $P_n(x)$ and an infinite series $Q_n(x)$ for $n = 0, 1, 2, 3, \cdots$. This polynomial solution[20] $P_n(x)$ is called the *Legendre function of the first kind of order* n. It is also known as the *Legendre polynomial of degree* n.

Continuing the discussion from the last paragraph, for each $n \in \mathbb{N}_0$, we obtain a pair of linearly independent solutions of the Legendre DE, one of which is a polynomial and the other an infinite power series which converges in $(-1, 1)$.

$$\begin{aligned}
n = 0, \quad & P_0(x) = 1 \\
& Q_0(x) = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots, \\
n = 1, \quad & P_1(x) = x \\
& Q_1(x) = 1 - x^2 - \frac{1}{3}x^4 + \cdots, \\
n = 2, \quad & P_2(x) = \frac{1}{2}(3x^2 - 1) \\
& Q_2(x) = x - \frac{2}{3}x^3 - \frac{1}{5}x^5 + \cdots, \\
n = 3, \quad & P_3(x) = \frac{1}{2}(5x^3 - 3x) \\
& Q_3(x) = 1 - 6x^2 - 3x^4 + \cdots, \\
n = 4, \quad & P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \\
\vdots \qquad & \quad \vdots
\end{aligned}$$

When the other linearly independant solution, which is an infinite series $Q_n(x)$, is normalised appropriately it is called the *Legendre function of the second kind*[17]. Q_n converges in the interval $(-1, 1)$ and diverges outside it. The Legendre function of second kind of degree n = 0 is given by:

$$\begin{aligned}
Q_0(x) &= x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots \\
&= \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).
\end{aligned}$$

This becomes unbounded as x tends to ± 1 from within the interval $(-1, 1)$. The same is true for $Q_1(x)$ and the other Legendre functions $Q_n(x)$[17]. Thus, the only eigenfunctions of Legendre's equation which are bounded at ± 1 are therefore the Legendre polynomials P_n. The first few Legendre polynomials are as follows:

$$\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= x, \\
P_2(x) &= \frac{1}{2}(3x^2 - 1), \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x), \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3), \\
P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x).
\end{aligned}$$

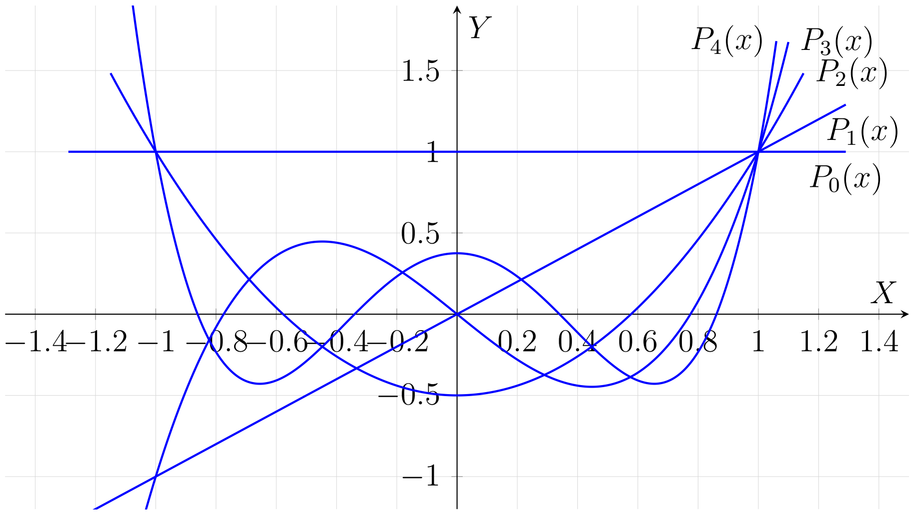


Figure 8: The first five Legendre polynomials

We have seen that only for certain endpoint conditions, the boundary term in (41) vanishes. Now we will consider those endpoint conditions for which this same boundary term vanishes in limit, that is,

$$\lim_{\alpha\to a,\beta\to b} \Big[p(x)[u(x)v'(x) - u'(x)v(x)]\Big]\Big|_{x=\alpha}^{x=\beta} = 0. \tag{42}$$

For example, the endpoint condition that $p(a) = p(b) = 0$ and the derivatives of the eigenfunctions be bounded on $[a, b]$, implies property (42). Therefore, for such endpoint conditions, using lemma(2.2.2) we get that,

$$(\lambda - \mu)\int_a^b \rho(x)u(x)v(x)dx = 0,$$

for any two square-integrable eigenfunctions $u(x)$ and $v(x)$, corresponding to eigenvalues λ and μ. Using this discussion and the Cauchy-Schwartz Inequality, we can present the following theorem for distinct eigenvalues.

Theorem 2.3.1. *Square-integrable eigenfunctions u and v corresponding to distinct eigenvalues of a singular SL system are orthogonal to each other with weight function ρ whenever the boundary term vanishes in limit, that is (42).*

Example. For a fixed m the Bessel DE:

$$[xu']' + \Big(k^2x - \frac{m^2}{x}\Big)u = 0,$$

in the interval $0 < x \leq a$ together with the endpoint condition that for a solution u, $u(a) = 0$ and $u(x)$ be bounded as $x \to a$ forms a singular SL system.

Firstly, we identify the point $x = 0$ as a *regular singular point*[22] and then employ the *Frobenius method*[22] of solving DEs. We get a solution of this DE as:

$$J_m(kx) = \sum_{l=0}^{\infty} \frac{(-1)^l k^{2l}}{(l!)(l+m)!}(x/2)^{2l+m}. \tag{43}$$

This particular solution (43) defines a function $J_m(kx)$ which is called the *Bessel function of first kind of order* m. Thus, the eigenfunctions of this singular SL system are the Bessel functions[2] $J_n(k_j x)$, where $k_j a$ is the j$^{\text{th}}$ zero of the Bessel function $J_n(x)$ of order n.

Bessel functions of the first kind of order $0, 1, 2$ with $k = 1$ are as follows:

$$\begin{aligned}
J_0(x) &= \sum_{l=0}^{\infty} \frac{(-1)^l}{(l!)^2}(x/2)^{2l},\\
J_1(x) &= \sum_{l=0}^{\infty} \frac{(-1)^l}{(l!)(l+1)!}(x/2)^{2l+1},\\
J_2(x) &= \sum_{l=0}^{\infty} \frac{(-1)^l}{(l!)(l+2)!}(x/2)^{2l+2}.
\end{aligned}$$

From theorem(2.3.1) it clearly follows that, the eigenfunctions of singular SL systems are also orthogonal, provided that they are square-integrable relative to the weight function ρ. A real-valued function f is said to be square integrable relative to the weight function[2] $\rho(x) > 0$ over an interval I, when $\int_I f^2(x)\rho(x)dx < 0$. Thus, on applying theorem(2.3.1) to the examples consisting the Legendre SL system and the Bessel SL system, we get the following conclusions:

1. The orthogonality of relation of the Legendre polynomials $P_n(x)$, that is, for $m \neq n$

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0. \tag{44}$$

2. The orthogonality of relation of the Bessel functions, that is, for $k_i \neq k_j$

$$\int_0^a xJ_n(k_i x)J_m(k_j x)dx = 0, \text{ if } J_n(k_i a) = J_m(k_j a) = 0. \tag{44}$$

Note that the orthogonality relation of eigenfunctions, mentioned in theorem(2.3.1), depends only on the endpoint condition, i.e., whether the endpoint condition implies the boundary term to vanish.

Example. We consider the Hermite DE:

$$u'' - 2xu' + \lambda u = 0, \quad -\infty < x < \infty.$$

Remark that the Hermite DE is not even an SL equation, because it is not self-adjoint. Hence, it cannot be used to form an SL system. However, an SL system can be formed using an equivalent Hermite SL equation : $y'' + [\lambda - (x^2 - 1)]y = 0, \quad -\infty < x < \infty$.

On solving, we obtain a polynomial solution of degree n for $\lambda_n = 2n$. On appropriately normalizing these, we get polynomials which are called the *Hermite polynomials*, denoted as $H_n(x)$.

The first few Hermite polynomials are as follows:

$$\begin{aligned} &H_0(x) = 1, \\ &H_1(x) = 2x, \\ &H_2(x) = 4x^2 - 2, \\ &H_3(x) = 8x^3 - 12x. \end{aligned}$$

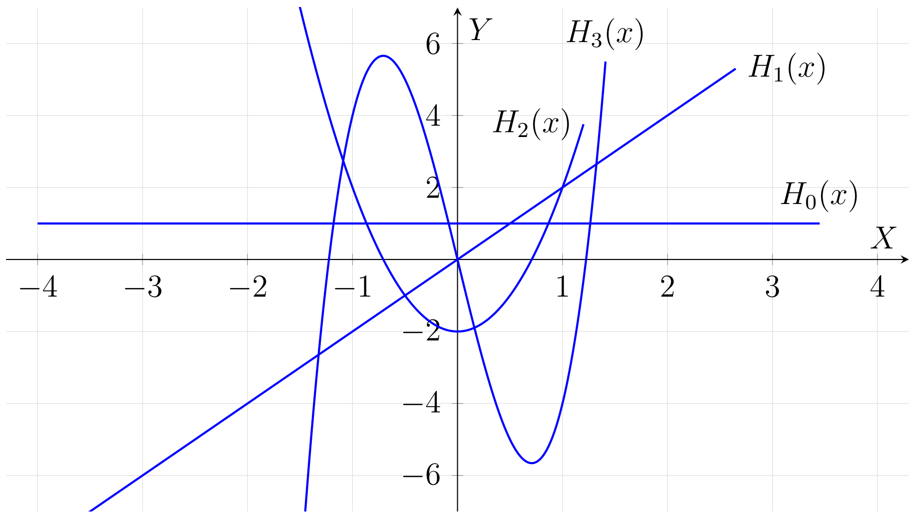


Figure 9: The first four Hermite polynomials

Forming any singular system using the Hermite DE will never result to an singular SL system. But here, the Hermite polynomials of degree n are square-integrable eigenfunctions and are also orthogonal with respect to the positive weight function e^{-x^2}, that is, for $m \neq n$, we have

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = 0. \tag{44}$$

2.4 Qualitative analysis on the zeroes of eigenfunctions

From the example produced in the beginning of section(2.2) it is evident that the two solutions of the regular SL system comprising the Helmholtz DE on $\mathbb{R}$ has an infinite sequence of alternating zeroes distributed uniformly, and are given as follows where the values in bold belongs to one eigenfunction and the rest belongs to another :

$$\cdots < -\frac{3\pi}{2} < \boldsymbol{-\pi} < -\frac{\pi}{2} < \mathbf{0} < \frac{\pi}{2} < \boldsymbol{\pi} < \frac{3\pi}{2} < \cdots .$$

We shall investigate and present results to answer whether such behaviour is completely accidental or not. We will also investigate the occurence that in a finite interval the number of zeroes of the eigenfunctions increases as their corresponding eigenvalue increases†.

For any fixed value of the parameter λ the Sturm-Liouville differential equation takes the following form (45). We will now go through a robust approach to study the solutions of a self-adjoint second order DE

$$\frac{d}{dx}\left[P(x)\frac{du}{dx}\right] + Q(x)u = 0; \qquad a < x < b \tag{45}$$

where $P(x) > 0$ is C^1 and Q is continuous.

Temporarily, we put aside the SL systems/ SL problems we were addressing and investigate equation(45) under the above considerations. For now, our primary objective is to comment on the number of zeroes in the interval $a < x < b$, of the solution of (45). Basically, we want to find out how frequently does the solutions of (45) oscillate on the interval under consideration, that is, the number of zeroes they have for $x \in (a, b)$.

2.4.1 Prüfer substitution

We use a slight modification of the Poincare's phase plane[2]. We perform the *Prüfer Substitution* in (45) as

$$P(x)u'(x) = r(x)\cos\theta(x); \quad u(x) = r(x)\sin\theta(x) \tag{46}$$

such that the newly introduced dependant variables r and θ are defined by the following formulas

$$r^2 = u^2 + (Pu')^2, \quad \theta = \arctan\left(\frac{u}{Pu'}\right) = \operatorname{arccot}\left(\frac{Pu'}{u}\right). \tag{47}$$

The variable r is called the *Amplitude* and θ the *Phase Variable*. It can be verified through some direct calculations that for non-zero r, the correspondences $(Pu', u) \rightleftharpoons (r, \theta)$ defined by (46) behave well so that global properties of a function like continuity and differentiability in one plane remains preserved analogously in the other plane † . Further, the uniqueness theorem[2] suggests that if $u(x) = u'(x) = 0$ for a certain x, then $u \equiv 0$, the trivial solution. As $P(x) > 0$, so from (47), r is non-zero for non-trivial solutions of (45).

Now we consider the relation (48) and on differentiating it we get

$$\cot\theta = \frac{Pu'}{u} \tag{48}$$

$$\implies -\csc^2(\theta)\cdot\frac{d\theta}{dx} = \frac{(Pu')'}{u} - \frac{P(u')^2}{u^2} = -Q(x) - \frac{1}{P(x)}\cot^2(\theta)$$

$$\implies \frac{d\theta}{dx} = Q(x)\sin^2(\theta) + \frac{1}{P(x)}\cos^2(\theta) = F(x,\theta). \tag{49}$$

† Remark: In the formula for θ in (47), we use the arctan formula when Γ is close to the Pu'-axis of the (Pu', u)-phase plane and the arccot formula when it is close to the u-axis.

Note that, in the case when $\theta \equiv 0 \mod (\pi)$ the above derivation of (49) from (48) will not work. However, in such case (49) can be derived by differentiating the relation $\tan\theta = u/Pu'$.

On differentiating $r^2 = (Pu')^2 + u^2$,

$$\begin{aligned}
2r \cdot \frac{dr}{dx} &= 2uu' + 2Pu'[P'u' + Pu''] \\
\Longrightarrow 2r \cdot \frac{dr}{dx} &= 2uu' - 2Pu'(Qu) \\
\Longrightarrow 2r \cdot \frac{dr}{dx} &= 2(Pu')\big[\frac{1}{P} - Q\big]u \\
\Longrightarrow \frac{dr}{dx} &= \big[\frac{1}{P} - Q\big]r\sin(\theta)\cos(\theta) = \frac{1}{2}\Big[\frac{1}{P} - Q\Big]r\sin(2\theta).
\end{aligned} \tag{50}$$

DE (49) and (50) together make a system of DEs and every non-trivial solution of this system defines a unique solution of the DE(45) by Prüfer substitution (46). In this sense, the system of DEs (49)-(50) and the DE(45) are equivalent.

The system (49)-(50) is called the *Prüfer System* associated with the self-adjoint DE (45).

DE(49) of the Prüfer system is a first-order DE in θ and x alone. Further, it also satisfies a Lipschitz condition (due to lemma(I.I.II)) with the Lipschitz constant L, such that

$$L = \sup_{a<x<b}\Big|\frac{\partial F}{\partial\theta}\Big| \leq \sup_{a<x<b}|Q(x)| + \sup_{a<x<b}\frac{1}{|P(x)|}.$$

Here, it is evident that the Lipschitz constant L is finite in any closed interval where the functions $P(x)$ and $Q(x)$ are continuous. Hence, the existence and uniqueness theorems[2] imply that DE (49) has a unique solution $\theta(x)$ for an initial value $\theta(a) = \gamma$, given that both $P(x)$ and $Q(x)$ are continuous at $x = a$.

Now, with a known $\theta(x)$ we can get $r(x)$ after a quadrature

$$r(x) = r(a)\cdot\exp\Big[\frac{1}{2}\int_a^x \big[\frac{1}{P(t)} - Q(t)\big]\sin(2\theta)dt\Big].$$

Thus, note that each solution of the Prüfer system (49)-(50) depends on two constants, $r(a)$ the *initial amplitude* and $\gamma = \theta(a)$ the *initial phase*. Further, due to (46), *changing the intial amplitude $r(a)$ just multiplies a solution $u(x)$ by a constant factor.* Therefore, we conclude that *the zeroes of any non-trivial solution of the linear second order self-adjoint DEs like (45) can be located by just studying the DE (49), which is a first order DE.*

2.4.2 The Sturm Comparison Theorem

Due to the way of Prüfer substituion is designed, we now know that the zeroes of any non-trivial solution $u(x)$ of DE(45) occurs only at those x where the phase function $\theta(x)$ in the Prüfer substitution done above, takes the values $0, \pm\pi, \pm 2\pi, \cdots$. Notice that, at each of these points $cos^2(\theta) = 1$, and by (49), $\frac{d\theta}{dx}$ is positive. *Geometrical interpretation* *of this is that, the curve Γ in the $(Pu', u)-$plane, corresponding to a non-trivial solution u of (45), will miss and rotate around the origin, and can cross the $Pu'-$axis for $\theta = n\pi$, that too counterclockwise.*

Equations of the form similar to DE(45), for which some of its solutions have two or more zeros in the open interval (a, b) will be called *oscillatory*. Note that in this definition, having one zero will not work as any equation of the form (45) has a solution with one zero. Primarily, our concern is towards the conditions which guarantees that equation (45) is *oscillatory.* The work presented in this section is based on the intention of getting an idea about the distribution of zeroes of non-trivial solutions of DE(45), which eventually should help us in knowing about their number of zeroes in a given interval.

Now, let us consider the following DE with the same form of DE(49), but having coefficients $Q_1(x) \geq Q(x)$ and $P_1(x) \leq P(x)$ from the same respective function classes, as:

$$\frac{d\theta}{dx} = Q_1(x)\sin^2(\theta) + \frac{1}{P_1(x)}\cos^2(\theta) = F_1(x, \theta). \tag{51}$$

Note that due to the choices of the coefficients in an interval I, $F_1(x, \theta) \geq F(x, \theta)$ in the same interval. Further, due to these considerations we also get a similar DE to DE(45) as,

$$\frac{d}{dx}\Big[P_1(x)\frac{du}{dx}\Big] + Q_1(x)u = 0; \quad a \leq x \leq b. \tag{52}$$

Assume $\theta_1(x)$ is a non-trivial solution of DE(51) and $\theta(x)$ is a non-trivial solution of DE(49), where $a, b \in \mathbb{R}$ are any two consecutive zeroes of $u(x)$. WLOG, say $\theta(a) = 0$ and $\theta(b) = \pi$. Due to the discussion done in the first paragraph of this subsection, which says that at integral multiples of π $d\theta/dx$ is positive, we know that $\theta(b)$ has to be π and not $-\pi$, given that $\theta(a) = 0$.

<u>CASE - I</u> Let us now first consider case when $\theta_1(a) > \theta(a)$. Due to corollary(I.I.V), we get the relation $\theta_1(x) > \theta(x)$ throughout the interval $[a, b]$. Now if $0 < \theta_1(a) < \pi$ along with $\pi < \theta_1(b)$, then due to continuity of $\theta_1(x)$ and by *Intermediate Value Theorem* we see that $\exists x^* \in (a, b) \ni \theta_1(x^*) = \pi$. This implies the occurence that *between any two zeroes of a non-trivial solution $u(x)$ of DE(45), there lies at least one zero of every non-trivial solution $u_1(x)$ of DE(52).* Further, when we have have $\pi < \theta_1(a)$ along with $\theta_1(b) < 2\pi$, we make a construction as $\theta_2 : [a, b] \to (0, \pi)$ such that $\theta_2(x) = \theta_1(x) - \pi$. Here again applying *IMVT* we get $x_0 \in (a, b)$ such that $\theta_2(x_0) = \theta(x_0)$. As for a $0 < h$, we have $\theta_2(x_0+h) < \theta(x_0+h)$, so clearly $\theta_1'(x_0) = \theta_2'(x_0) < \theta'(x_0)$, which is a major contradiction. By this, we exhaust all considerations under this case.

<u>CASE - II</u> We now consider $\theta_1(a) = \theta(a)$. Using Comparison Theorem(I.I.IV), we get that in $[a, b]$ $\theta_1(x) \geq \theta(x)$. At the right endpoint, the relation $\theta_1(b) \geq \theta(b)$ again leads to the occurence mentioned above. Next is the subcase when $\theta_1(b) = \theta(b)$. For this, we claim that $\theta_1(x) \equiv \theta(x)$ in $[a, b]$. On the contrary let us assume a point $x_0 \in (a, b) \ni \theta_1(x_0) > \theta(x)$. Again using corollary(I.I.V) on the interval (x_0, b) we get $\theta_1(b) > \theta(b)$, a contradiction. This proves our claim, and we can see that in this scenario the mentioned occurence does not happen. Observe that $\theta_1(x) \equiv \theta(x)$ in $[a, b]$ if and only if we have $u(x) \equiv c \cdot u_1(x)$, it follows that the above mentioned occurence about the zeroes of $u(x)$ and $u_1(x)$ does not happen whenever we have $u(x) \equiv c \cdot u_1(x)$.

Proceeding ahead, now we go through different cases for the condition $\theta_1(x) \equiv \theta(x)$. For the first case, when $\theta(x) \equiv \theta_1(x)$ is neither an integer multiple of π nor of $\pi/2$, this condition leads to $P_1 \equiv P$ and $Q_1 \equiv Q$. The second case when $\theta(x) \equiv \theta_1(x)$ is an integer multiple of π, is ruled out as it vacously implies that both $u(x)$ and $u_1(x)$ are trivial solutions of the relevant DEs. Finally in the third case when $\theta(x) \equiv \theta_1(x)$ is an odd-integer multiple of $\pi/2$, we only get $Q_1 \equiv Q$ and hence we cannot confirm whether $u(x) \equiv c \cdot u_1(x)$ holds or not. These discussions completes the proof of a well-celebrated result, which we present as the following theorem.

Theorem 2.4.1. ***Sturm Comparison Theorem*** *Let $P(x) \geq P_1(x) > 0$ and $Q_1(x) \geq Q(x)$ in the DEs*

$$\frac{d}{dx}\left(P(x)\frac{du}{dx}\right) + Q(x)u = 0, \qquad \frac{d}{dx}\left(P_1(x)\frac{du_1}{dx}\right) + Q_1(x)u_1 = 0.$$

Then, between any two zeroes of a non-trivial solution $u(x)$ of the first DE, there lies at least one zero of every real solution of the second DE, except when $u(x) \equiv c \cdot u_1(x)$. Further, this occurence does not happen when $P_1 \equiv P$ and $Q_1 \equiv Q$, except possibly in intervals where $Q_1 \equiv Q \equiv 0$.

The Sturm Separation Theorem(I.II.I), which says about the alternating behaviour of zeroes of linearly independent solutions of DE(45), follows as a corollary, by comparing two linearly independent solutions of the same DE. Although, the proof given in Appendix(I.II) doesn't imitate the arguments made in this section. In the context of SL equations, since weight functions $\rho(x) > 0$, $Q(x) \equiv Q_1(x)$ clearly implies that $\lambda = \lambda_1$. Therefore, instead of considering two different self-adjoint DEs we can present the following lemma concerning a single SL equation.

Lemma 2.4.2. *When $u(x)$ and $v(x)$ are eigenfunctions of an SL system corresponding to distinct eigenvalues λ and μ respectively. If $\mu \geq \lambda$, then between any two zeroes of $u(x)$, there lies at least one zero of $v(x)$.*

Example. Let us revisit the example discussed in the beginning of section(2.2), where $\sin(nx)$ and $\cos(nx)$ were two eigenfunctions for eigenvalue $\lambda = n^2$, squares of integers. Figure(10) below visualises these eigenfunctions for $\lambda_1 = 1$ and $\lambda_4 = 16$. It can be seen that between any two zeroes of the *blue* curves, there lies atleast one zero of their corresponding *orange* curves, theorem(2.4.2) in effect.

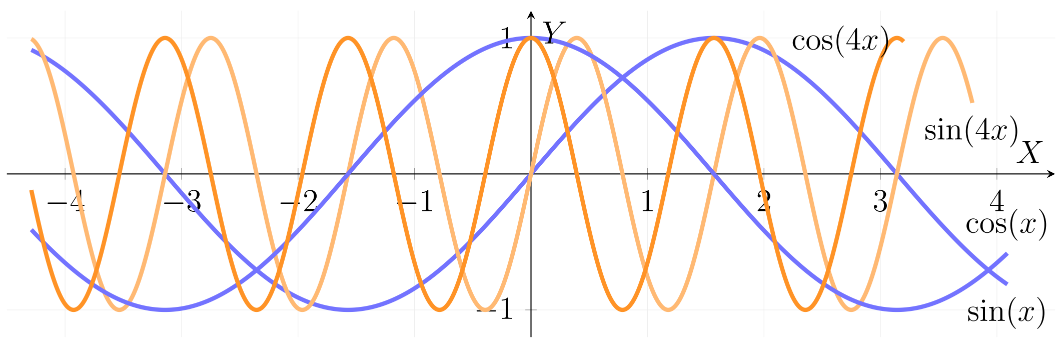


Figure 10: An example to visualize Sturm-Comparison Theorem(2.4.2)

Using these arguments, we can deduce a result concerning the number of maxima/minima of non-trivial solutions $u(x)$ of DE(45). For self-adjoint DEs (45), whenever $\theta = (2n+1)\frac{\pi}{2}$, we have seen above that the condition $Q(x) > 0$ implies that $\frac{d\theta}{dx} > 0$. Since, $P(x) > 0$, from (49) it is clear that, $\theta = (2n+1)\frac{\pi}{2}$, that is, $\cos\theta = 0$ iff $u'(x) = 0$. It follows that, *if $Q(x) > 0$, then any non-trivial solution of (46) has exactly one local maxima or local minima between any two of its consecutive zeroes.*

Definition 2.4.3. ***Isolated zeroes*** *A function $f : I \to \mathbb{R}$ is said to have an isolated zero at $x_0 \in I$, if $f(x_0) = 0$ and there exists a neighborhood U of x_0 such that $f(x) \neq 0$ for all $x \in I \cap U \setminus \{x_0\}$.*

Remark that, in the second subcase of discussions done in the paragraph preceeding Theorem(2.4.1), we find $\frac{d\theta}{dx}$ to be positive, whenever $\sin\theta(x)$ is zero. So, by continuity of sin and θ, it follows that in a neighborhood of such points, $\theta(x)$ is strictly increasing. Therefore, we can also conclude that non-trivial solutions of self-adjoint DEs (45), or in particular eigenfunctions of SL equations only have *isolated zeroes.*

Before proceeding ahead for the next section, note that another major takeaway from the discussions made in this section, which can be easily remembered although might be a bit imprecise is that, *as Q increases and P decreases, the number of zeros of every non-trivial solution of DEs like (45) increases.* Figure(10) and the example above, in a way, does a fair advocacy of this result.

2.4.3 Sturm Oscillation Theorem

Now that we have considered the zeroes of solutions of general self-adjoint DEs, we now try to explore the variations of the number of zeroes of the eigenfunctions of a regular SL system (38)-(39), with its eigenvalue λ.

There by, setting $P(x) = p(x)$ and $Q(x) = \lambda\rho(x) - q(x)$, in (39) we obtain (45). Since $u = 0$ iff $\sin\theta = 0$ in the Prüfer substitution (46), the zeroes of any solution of (38) are the points for which $\theta(x) = 0, \pm\pi, \pm2\pi, \pm3\pi, \cdots, \pm n\pi$, such that θ is a solution of the associated Prüfer equation:

$$\frac{d\theta}{dx} = \big[\lambda\rho(x) - q(x)\big]\sin^2\theta + \frac{1}{p(x)}\cos^2\theta, \qquad a \leq x \leq b. \tag{53}$$

As we now consider our SL system to be regular, here $p(x) > 0$, $\rho(x) > 0$ $\forall x \in [a, b]$.

Let $\theta(x, \lambda)$ be the solution of (53) which satisfies the initial condition $\theta(a, \lambda) = \gamma$, $\forall\lambda$ and for a fixed γ. Here, due to the Prüfer substitution, any fixed γ is determined by

$$\tan\gamma = \frac{u(a)}{p(a)u'(a)} = \frac{-\alpha'}{p(a)\alpha}, \qquad 0 \leq \gamma < \pi.$$

Note that, the constants α and α' come from the initial condition (separated endpoint condition) $\alpha u(a) + \alpha' u'(a) = 0$. For a fixed γ, we shall now explore the behavior of the function $\theta(x, \lambda)$ for $-\infty < \lambda < \infty$ on the domain $[a, b]$. Due to the Corollary(I.I.VI) or the comparison theorems presented in the Appendix(I.I), we get the following lemma.

Lemma 2.4.4. *For a fixed $x > a$, $\theta(x, \lambda)$ is a strictly increasing function of the variable λ.*

Proof. Let $\lambda_1 < \lambda_2$, and arbitrarily we choose and fix an $x_1 > a$. We consider the following two DEs in $[a, b]$

$$\frac{d\theta}{dx} = \underbrace{[\lambda_1 \rho(x) - q(x)] \sin^2\theta + \frac{1}{p(x)} \cos^2\theta,}_{=F(x,\theta)}$$

$$\frac{d\theta}{dx} = \underbrace{[\lambda_2 \rho(x) - q(x)] \sin^2\theta + \frac{1}{p(x)} \cos^2\theta,}_{=G(x,\theta)}$$

along with the initial conditions $\theta(a, \lambda_i) = \gamma$, for $i = 1$ and 2 respectively, such that $0 \leq \gamma < \pi$. As we are under the consideration of regular SL systems, we have $\rho(x) > 0$. Due to this, here we get $F(x, \theta) < G(x, \theta)$. Thus using Corollary(I.I.VI) we get that either $\theta(x_1, \lambda_1) < \theta(x_1, \lambda_2)$ or $\theta(x, \lambda_1) \equiv \theta(x, \lambda_2)$ $\forall x \in [a, x_1]$. The later scenario is a contradiction as distinct I-order ODEs cannot have identical solutions. Therefore, this completes the proof. $\square$

Now say x_n is a point in the domain such that $\theta(x_n, \lambda) = n\pi$, and it follows that from DE(53) that

$$\left.\frac{d\theta(x, \lambda)}{dx}\right|_{x=x_n} = \frac{1}{p(x_n)} > 0.$$

Hence for a fixed λ, $\theta(x, \lambda)$, a function in x, is increasing where it crosses the line $\theta = n\pi$. Thus, $\theta(x, \lambda)$ stays above the line $\theta = n\pi$ for $x > x_n$. The result concluded from this discussion can be presented as the following lemma.

Lemma 2.4.5. *Suppose that for some $x_n > a$, $\theta(x_n, \lambda) = n\pi$, where $n \in \mathbb{Z}_{\geq 0}$. Then $\theta(x, \lambda) > n\pi$ forall $x > x_n$.*

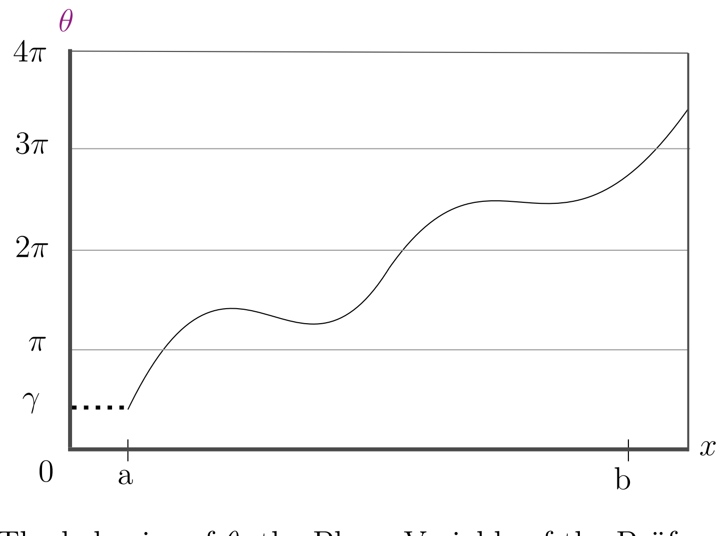


Figure 11: The behavior of θ, the Phase Variable of the Prüfer substitution.

Figure(11) depicts the behavior of $\theta(x, \lambda)$, the Phase Variable of the Prüfer substitution. It is not in general a monotonically increasing function, but once it reaches the value $n\pi$ at a certain point x_n, with accordance to lemma(2.4.5) it remains greater than $n\pi$ for $x > x_n$. We define $x_n(\lambda)$ to be the smallest of all x such that $\theta(x, \lambda) = n\pi$. We wish to see the well-definedness of $x_n(\lambda)$ for a fixed n but a sufficiently large enough λ.

Note that the coefficient function $q(x)$, $p(x)$ and $\rho(x)$ are all C^0 functions and since we are considering a regular SL system, we are working on a compact interval, say $[a, b]$. So, let q_M and p_M be the maxima of $q(x)$ and $p(x)$, respectively, and let ρ_m be the minimum of $\rho(x)$ for $a \leq x \leq b$. On solving the following II-order constant coefficient ODE :

$$p_M u'' + (\lambda\rho_m - q_M)u = 0, \qquad \lambda > \frac{q_M}{\rho_m}, \tag{54}$$

we get that for $k^2 = (\lambda\rho_m - q_M)/p_M$ the function $\sin kx$ is a solution of (54). On translating this function by a we get $u_1(x) = \sin k(x - a)$, which is again a solution of (54) and also has a root at $x = a$. Here, it is easy to see that successive zeroes of $u_1(x)$ are all distributed uniformly with a distance of $\pi/k = \pi\sqrt{p_m/(\lambda\rho_m - q_M)}$ from each other. By the Sturm Comparison Theorem(2.4.1), it follows that any non-trivial solution $u(x)$ of the SL equation(38) must have at least one zero between any two zeroes of the function $u_1(x)$. Proceeding a calculation starting with the relation $n\pi/k < b - a$ will fetch us $\lambda_{min} = \frac{p_M}{\rho_m}[\frac{n\pi}{b-a}]^2 + \frac{q_M}{\rho_m}$, the minimum value which λ must exceed so that $u_1(x)$ has atleast n zeroes in (a, b). Since it is now established that, for any given $n \in \mathbb{N}$, $u_1(x)$ has n zeroes on (a, b) when λ is sufficiently large, it follows that $u(x)$ also has at least n zeroes in (a, b) for sufficiently large λ. Therefore, $\theta(x, \lambda)$ admits the value $n\pi$ for sufficiently large λ. Thus, $\theta(x, \lambda)$ being a continuous function in both x and λ, must take all values between $\theta(a, \lambda) = \gamma < \pi$ and $n\pi$. Hereby, we conclude that $x_n(\lambda)$ is well-defined for sufficiently large λ.

After using Theorem(I.III.II) in Appendix(I.III), we find $\theta(x, \lambda)$ to be continuous in both x and λ for $a \leq x \leq b$ and $-\infty < \lambda < \infty$. Now that it is established that $\theta(x, \lambda)$ is well-defined even for sufficiently large λ, we use Lemma(2.4.4). It tells us that $\theta(x, \lambda)$ is a strictly increasing function of λ. It suffices to conclude that, $x_n(\lambda)$ is a monotonically decreasing function of λ.

Further, observe that the point $x_n(\lambda)$ for $u(x)$ is located between $(n-1)^{\text{th}}$ and n^{th} zero of $u_1(x)$. As both of these zeroes of $u_1(x)$ tend to a as λ tends to ∞, we conclude that $x_n(\lambda)$ tends to a as λ tends to ∞. All these discussions completes the proof for the following lemma.

Lemma 2.4.6. *For a given fixed $n \in \mathbb{Z}_{>0}$, and sufficiently large λ, the function $x_n(\lambda)$ is defined and continuous. Further, it is a decreasing function of λ and we have*

$$\lim_{\lambda\to\infty} x_n(\lambda) = a.$$

Due to Lemma(2.4.6) above, we now know that even for a sufficiently large λ, we have the smallest number $x_n < x$ such that $\theta(x, \lambda) = n\pi$. Note that this happens for every positive integer, i.e., it is thus true for all $n \in \mathbb{Z}_{>0}$. Here, we now use the result from lemma(2.4.6) that $\lim_{\lambda\to\infty} x_n(\lambda) = a$ and lemma(2.4.5), to deduce that *for a fixed $x > a$, $\theta(x, \lambda) \to \infty$ as $\lambda \to \infty$.*

The next natural step is to comment on the behaviour of $\theta(x, \lambda)$ when $\lambda \to -\infty$. Let us now consider $\gamma < \gamma_1 < \pi$ and $\epsilon < 0$. For $x_1 \in (a, b]$, the slope of the segment in $x\theta-$plane formed by joining the points (a, γ_1) and x_1, ϵ equals $(\epsilon - \gamma_1)/(x_1 - a)$. Observe that this slope is finite and hence, for any sufficiently large negative λ and for a point (x, θ) on this segment, the slope of $\theta(x, \lambda)$,as given by (53) will be less than the slope of the segment. It follows that, for sufficiently large negative λ, the function $\theta(x, \lambda)$ will lie below the segment for $a \leq x \leq x$. By this, we can conclude that $\theta(x_1, \lambda) < \epsilon$ for sufficiently large negative λ. Recall that, using the arguments used as a proof for lemma (2.4.5), $\theta(x_1, \lambda) > 0$. Hence, it follows that $|\theta(x_1, \lambda)| < \epsilon$. Since, $x_1 \in (a, b]$ and $\epsilon > 0$ were both chosen arbitrarily, we can use these above arguments along with lemmas (2.4.4), (2.4.5) and (2.4.6) as a complete proof of the well-celebrated *Sturm Oscillation Theorem*, which is stated as follows.

Theorem 2.4.7. ***Sturm Oscillation Theorem*** *The solution $\theta(x, \lambda)$ of the DE(53) satisfying the initial condition $\theta(a, \lambda) = \gamma$, $0 \leq \gamma < \pi$ for each λ, is a continuous and strictly increasing function of λ for any fixed $x \in (a, b]$. Moreover, for $a < x \leq b$ we have*

$$\lim_{\lambda \to \infty} \theta(x, \lambda) = \infty \qquad \text{and} \qquad \lim_{\lambda \to -\infty} \theta(x, \lambda) = 0.$$

Due to the Sturm Oscillation Theorem (2.4.7) we now know that in a finite interval, any non-trivial solution of a SL equation will have an incresing frequency of oscillation, thus increasing number of zeroes for an increasing value of λ. Once again, figure(10) and its corresponding example does a fair visual advocacy of this takeaway. Proceeding ahead, we know wish to have an estimate on the positions of the zeroes of a non-trivial solution of a regular SL equation (38), by comparing it with (54) and

$$p_m u'' + (\lambda \rho_M - q_m) u = 0, \tag{55}$$

where p_m and q_m are the minima of $p(x)$ and $q(x)$ respectively, and ρ_M the maxima of $\rho(x)$ for $x \in [a, b]$.

Let us now consider solutions of (54) and (55) with initial condition $u(a)/p(a)u'(a) = \tan \gamma$. Through inspection, we get their zeroes to be

$$a + \frac{n\pi - \gamma}{\sqrt{\lambda \rho_m - q_M / p_M}} \qquad \text{and} \qquad a + \frac{n\pi - \gamma}{\sqrt{\lambda \rho_M - q_m / p_m}}$$

respectively. Applying the Sturm Comparison Theorem(2.4.1) twice, we complete the proof of a result which we present as the following corollary.

Corollary 2.4.8. *For each $n \in \mathbb{Z}_{>0}$, let x_n be the n^{th} zero of the non-trivial solution of the SL equation(38). Then we have the following estimate on the position of x_n :*

$$\sqrt{\frac{p_m}{\lambda \rho_M - q_m}} \leq \frac{x_n - a}{n\pi - \gamma} \leq \sqrt{\frac{p_M}{\lambda \rho_m - q_M}}.$$

---

† Remark: All the qualitative analysis done in the span of Section(2.4) is also celebrated by the name of *Oscillation Theory*. Sources [8], [26] provides a good survey of important applications of Prüfer substitution and also provides a robust presentation of Oscillation Theory.

2.4.4 Sturm Convexity Theorem - an application

In this section we present and discuss the *Sturm Convexity Theorem*, which is one of many significant applications of the Sturm Oscillation Theory, in particular the Sturm-Comparison Theorem(2.4.1) we have discussed above.

Before we dive directly into the mentioned result, let us state this following result which will be used to complete the proof of the main result of this section. Note that the proof of the following theorem has been skipped here, but it can be found at [12], [27], which itself uses Sturm Comparison Theorem(2.4.1).

Theorem 2.4.9. *Let $y'' + Q(x)y = 0$ be a second-order DE, with $Q(x)$ being continuous in (a, b). Let $y(x)$ be a non-trivial solution of this DE in (a, b). Let $x_k < x_{k+1} < \cdots$ denote consecutive zeros of $y(x)$ in (a, b) arranged in an increasing order. Then*

1. *If $\exists\ Q_M > 0$ such that $Q(x) < Q_M$ in (a, b) then*

$$\Delta x_k = x_{k+1} - x_k > \frac{\pi}{\sqrt{Q_M}}.$$

2. *If $\exists\ Q_m > 0$ such that $Q(x) > Q_m$ in (a, b) then*

$$\Delta x_k = x_{k+1} - x_k < \frac{\pi}{\sqrt{Q_m}}.$$

Now, similar to the discussions in the section of Sturm Comparison Theorem let us consider the DE $u'' + Q(x)u = 0$ in an interval (a, b), where Q is continuous. Further, say $u(x)$ is a non-trivial solution of this DE, and $x_1 < x_2 < \cdots < x_k < x_{k+1} < x_{k+2} < \cdots$ are its consecutive roots in (a, b).

We proceed ahead with the assumption that Q is strictly increasing and is positive in (a, b). So, in the interval (x_k, x_{k+1}) we have $Q(x) < Q(x_{k+1})$, and in the interval (x_{k+1}, x_{k+2}) we have $Q(x) > Q(x_{k+1})$. Deploying Theorem(2.4.9), we get

$$\Delta x_k = x_{K+1} - x_k > \frac{\pi}{\sqrt{Q(x_{k+1})}}, \qquad \text{and} \qquad \Delta x_{k+1} = x_{k+2} - x_{k+1} < \frac{\pi}{\sqrt{Q(x_{k+1})}}.$$

From here, it is follows that $x_{k+2} - x_{k+1} < x_{k+1} - x_k$. Similarly, using analogous arguments and usage of theorem(2.4.1), it can be proven that for Q strictly decreasing instead of being strictly increasing, we get $x_{k+2} - x_{k+1} > x_{k+1} - x_k$.

This discussion leads to the proof of a specific case, which when generalised by dropping the assumption that Q is positive, leads to become the Sturm Convexity Theorem(2.4.10).

Theorem 2.4.10. *Let $u'' + Q(x)u = 0$ in (a, b) be a second order DE in normal form, such that $Q(x)$ is conitnuous in (a, b). Let $u(x)$ be a non-trivial solution of this DE with $x_1 < x_2 < \cdots < x_k < x_{k+1} < x_{k+2} < \cdots$ as its roots in (a, b). Then*

1. *If $Q(x)$ is strictly increasing in (a, b) then $x_{k+2} - x_{k+1} < x_{k+1} - x_k$.*
2. *If $Q(x)$ is strictly decreasing in (a, b) then $x_{k+2} - x_{k+1} > x_{k+1} - x_k$.*

In summary, the major takeaway from Sturm Convexity Theorem(2.4.10) is that for SL equation $u'' + Q(x)u = 0$ in the normal form, when compared with $u'' + m^2u = 0$ $u'' + M^2u = 0$ in (a, b) such that Q is continuous and $m^2 < Q(x) < M^2$ for $0 < m < M$, the zeroes $x_1 < x_2 < x_3 < \cdots$ of a non-trivial solution $u(x)$ satisfy

$$\frac{\pi}{M} < x_{i+1} - x_i < \frac{\pi}{m}, \qquad i = 1, 2, 3, \cdots .$$

CHAPTER 3: STURM-LIOUVILLE SYSTEMS - II

3.1 Introduction

After ending some necessary and crucial discussions on SL systems in the last chapter we now intend to dive deep into the theory of SL systems. Here onwards, our primary aim would be to show that eigenfunctions of any regular SL systems form a basis of appropriate L^2 spaces. In this chapter, we will present and discuss topics which will eventually lead to and aid us to establish our primary aim.

3.2 Sequence of eigenvalues

Till now, throughout all the examples of SL systems discussed, we have considered discrete eigenvalues. Further, due to the contents in Section(2.4), we now know the behaviour of eigenfunctions even for sufficiently large eigenvalues λ. Now in this section, we will explore the existence of an infinite series of eigenvalues and, consequently, sequence of eigenfunctions of a regular SL system having the following separated endpoint conditions

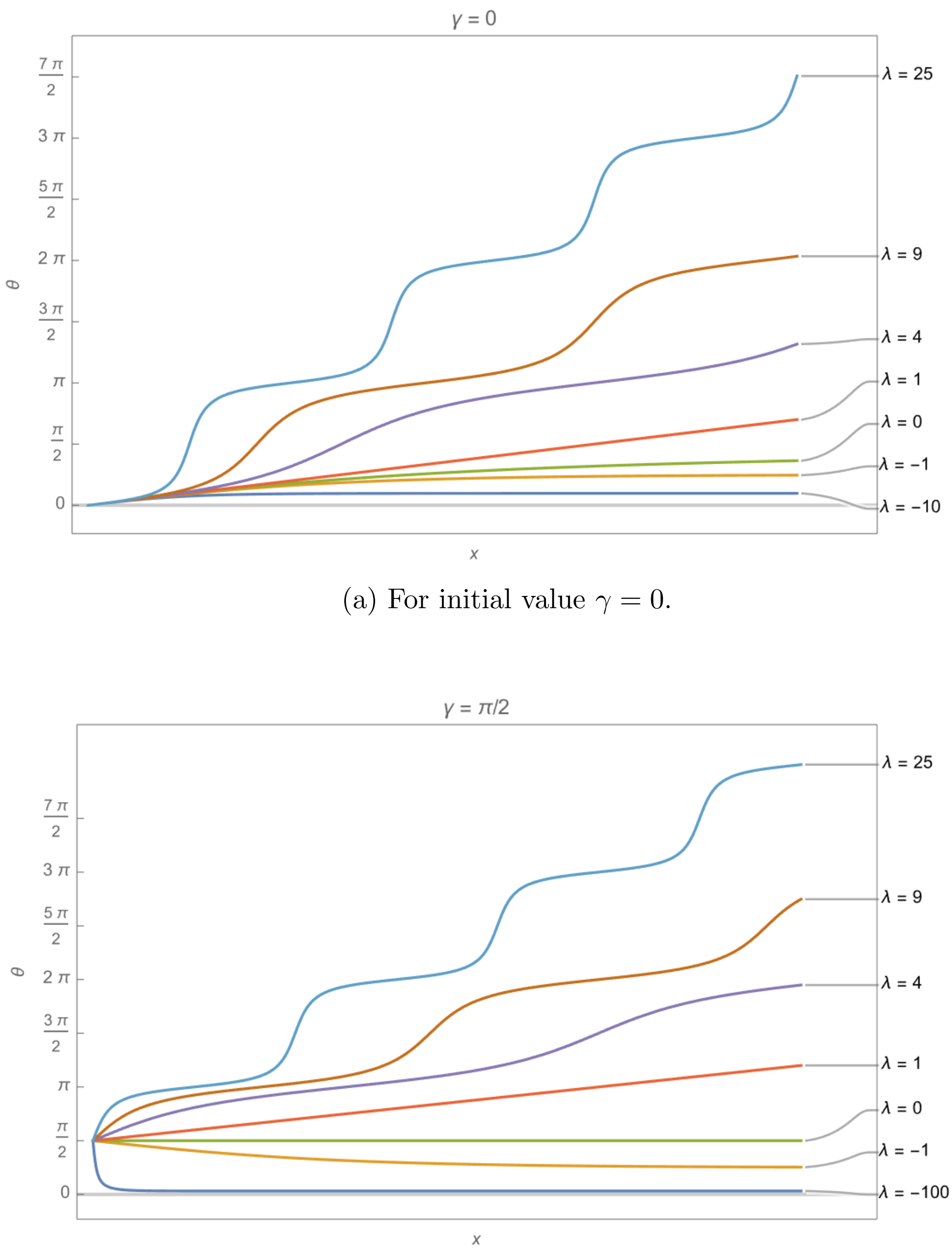
$$A[u] = \alpha u(a) + \alpha' u'(a) = 0, \qquad B[u] = \beta u(a) + \beta' u'(a) = 0. \tag{56}$$

We begin by transforming the endpoint conditions (56) into equivalent endpoint conditions for the phase function $\theta(x, \lambda)$ of the Prüfer system (49)-(50), associated with the DE(38). For $\alpha \neq 0$, the phase function $\theta(x, \lambda)$ must satisfy the initial condition $\theta(a, \lambda) = \gamma$ such that γ is the smallest non-negative number so that $0 \leq \gamma < \pi$, where $p(a) \tan \gamma = -\alpha'/\alpha$. For $\alpha = 0$, we choose $\gamma = \pi/2$. Similarly, we select $0 < \delta \leq \pi$ such that $p(b) \tan \delta = -\beta'/\beta$. By this, we have the achieved the equivalence we were trying to establish. *A solution $u(x)$ of the DE (38) for $a \leq x \leq b$ is an eigenfunction of the regular SL system obtained by imposing the endpoint condition (56), if and only if, for the corresponding phase function defined by (46), we have*

$$\begin{aligned} &\theta(a, \lambda) = \gamma, \qquad &&\theta(b, \lambda) = \delta + n\pi, \quad n = 0, 1, 2, 3, \cdots \\ &0 \leq \gamma < \pi, &&0 < \delta \leq \pi. \end{aligned} \tag{57}$$

It is clear that, any value of λ for which conditions (57) are satisfied, is an eigenvalue of the given regular SL system, and conversely. Let $\theta(x, \lambda)$ be the solution of (53) for the initial condition $\theta(a, \lambda) = \gamma$.

Example. Plots in Figure(12) shows the graph of the phase function $\theta(x, \lambda)$ corresponding to the DE $u'' + \lambda u = 0$ for various values of the parameter λ. One can inspect and find that for all λ, the slope of the graphs for $\theta \equiv 0 (mod\ \pi)$ is $1/p(x)$ and for all other values of θ, the slope of the graphs tends to infinity with λ. The waviness of the graphs hence expresses the fact that $1/P(x)$ in (49) for $u'' + \lambda u = 0$, is independant of the parameter λ.



(a) For initial value $\gamma = 0$.

(b) For initial value $\gamma = \pi/2$.

Figure 12: Plot for $\theta(x, \lambda)$ corresponding to $u'' + \lambda u = 0$, for varying λ ‡ .

‡ Figure(12) was generated by plotting the solutions of the corresponding parametric DE with parameter λ using Wolfram Mathematica. The entire process is due to computational methods and the code for the same can be found at the GitHub Repo with the link - https://github.com/Shubhajit412/THEORETICAL-AND-COMPUTATIONAL-CONSIDERATIONS-OF-STURM-LIOUVILLE-SYSTEMS.

Due to Lemma(2.4.4), $\theta(b, \lambda)$ is an increasing function of λ. Further due to Lemma(2.4.5), we have $\theta(b, \lambda) > 0$. So as λ increases from $-\infty$, there exists a first eigenvalue λ_0, for which the second condition in (57) gets satisfied, i.e., we have $\theta(b, \lambda_0) = \delta$. Gradually, as λ increases, there is an infinite sequence $\{\lambda_n\}$ for which the second boundary condition in (57) is satisfied, i.e., $\theta(b, \lambda_n) = \delta + n\pi$, for some given $n \in \mathbb{N}$. Note that each of this eigenvalues λ_n gives an eigenfunction

$$u_n(x) = r_n(x) \sin \theta(x, \lambda_n) \tag{58}$$

of the concerned SL system. Moreover, due to Sturm Oscillation Theorem(2.4.7), the eigenfunctions corresponding to λ_n, has exactly n zeros in the interval (a, b).

Also by Uniquesness Theorem for II-order DE [2], we know that, any two solutions of the DE(38) satisfying the same initial condition $\alpha u(a) + \alpha' u'(a) = 0$ are linearly dependant. Hereby, the conclusion of these discussions can be presented in the form of the following theorem.

Theorem 3.2.1. *Any regular SL system has an infinite sequence of real eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ with $\lim_{n\to\infty} \lambda_n = \infty$. The eigenfunction $u_n(x)$ belonging to the eigenvalue λ_n has exactly n zeros in the interval (a, b) and is uniquely determined upto multiplication by a constant.*

3.3 The Liouville Normal Form

We consider regular SL equation(38) and wish to simplify it considerably, through the following changes in the dependant and independent variables

$$u = y(x)w, \qquad t = \int h(x)\, dx; \qquad y > 0,\ h > 0. \tag{59}$$

Notice that if the functions y and h are positive and continuous in the given interval, then the first substitution in (59) leaves the location of zeros of any non-trivial solution unchanged, while the second one distorts the range of the independent variable. As the position of zeroes does not change, from Corollary(2.4.8) we use the inferrence that the bounds on the position of zeroes are independent of the substitution (59), thus the substitutions leaves the number of zeros of a solution in corresponding intervals unchanged.

From the second equation in (59), we obtain the identity $d/dx = h(x)d/dt$, and then use it obtain an equivalent DE to (38) in w and t. After substituting the above identity in DE(38), we get

$$\begin{aligned} h[hp(yw)_t]_t + (\lambda\rho - q)yw &= 0 \\ h\{pyhw_{tt} + [(hp)_t y + 2hpy_t]w_t + (hpy_t)_t w\} + (\lambda\rho - q)yw &= 0. \end{aligned}$$

Here the notation $[\cdot]_t$ is used to represent $d[\cdot]/dt$. Now we divide the last equation by pyh^2, the coefficient of the term w_{tt} to obtain the following DE equivalent to DE(38) when $h, y \in C^2$.

$$w_{tt} + \frac{1}{pyh}[(hp)_t y + 2hpy_t]w_t + \left[\frac{1}{pyh}(hpy_t)_t + \frac{1}{h^2 p}(\lambda\rho - q)\right]w = 0. \tag{60}$$

In equation (60), the term $\lambda(\rho/ph^2)w$ can be reduced to λw iff $h^2 = \rho/p$. Also observe that, the coefficient of w_t goes to zero iff $(hp)_t/hp = -2y_t/y$, which can be implied from the relation $y^2 = 1/hp$. Therefore, a much simplified equivalent version of DE(38) in w and t can be deduced by considering the following choices

$$u = w/\sqrt[4]{p(x)\rho(x)}, \qquad t = \int \sqrt{\rho(x)/p(x)}\ dx. \tag{61}$$

We call substitution (61) as the *Liouville's substitution*, due to which we can reduce DE(38) to *Liouville normal form*. As by definition of a regular SL equation, we have both $p(x) > 0$ and $\rho(x)$ throughout the working interval, substitution (61) makes $h(x)$ and $y(x)$ postive and C^2, whenever p and ρ are C^2. We can present the following theorem, keeping the above discussion as a mathematically legit evidence.

Theorem 3.3.1. *Liouville's substitution (61) transforms the regular SL equation (38) with coefficient functions $p,\ \rho \in C^2$ and $q \in C$ into the Liouville normal form*

$$\frac{d^2w}{dt^2} + [\lambda - \hat{q}(t)]w = 0, \tag{62}$$

where

$$\hat{q} = \frac{q}{\rho} + \frac{1}{\sqrt[4]{p\rho}} \cdot \frac{d^2}{dt^2}[\sqrt[4]{p\rho}]. \tag{63}$$

We can deduce the following alternative rational form of $\hat{q}$ by evaluating the second derivative in (63) using the identity $d/dt = (p/\rho)^{1/2}d/dx$

$$\hat{q} = \frac{q}{\rho} + \frac{p}{4\rho}\left[\left(\frac{p'}{p}\right)' + \left(\frac{\rho'}{\rho}\right)' + \frac{3}{4}\left(\frac{p'}{p}\right)^2 + \frac{1}{2}\left(\frac{p'}{p}\right)\left(\frac{\rho'}{\rho}\right) - \frac{1}{4}\left(\frac{\rho'}{\rho}\right)^2\right]. \tag{64}$$

Moreover, if DE (38) is defined on an interval $[a, b)$, and t is given by the definite integral $\int_a^x \sqrt{\rho(s)/p(s)}\ ds$ then the equivalent DE(62) is defined in the interval $[0, c)$ such that $\int_a^b \sqrt{\rho(x)/p(x)}\ dx$. Therefore, an SL equation (38) with $p, \rho \in C^2$ and $q \in C$ is transformed by Liouville's substitution into an SL equation(62) with $\hat{q} \in C$, as the dinominator in (63) remains bounded away from 0. Hence, we can formulate and present the following corollary.

Corollary 3.3.2. *Liouville's reduction (61) transforms regular SL systems again into regular SL systems, separated and periodic boundary conditions again into separated and periodic boundary conditions. Further, the transformed system has the same eigenvalues as the original system.*

An important question which must be answered is, *how does the above introduced Liouville's transformation affect the orthogonality of eigenfunctions?* Let $u(x)$ and $v(x)$ be transformed into the functions $f(x)$ and $g(x)$ by Liouville's reduction (61). Using the relevant identities from the discussions above, we get

$$\int_0^c f(t)g(t)dt = \int_a^b u(x)v(x)\sqrt{p(x)\rho(x)}\sqrt{\frac{\rho(x)}{p(x)}}dx = \int_a^b u(x)v(x)\rho(x)dx. \tag{65}$$

Hence, using the above identity (65), we infer the following corollary.

Corollary 3.3.3. *Liouville's reduction (61) transforms functions orthogonal with weight $\rho(x)$ again into orthogonal functions but with unit weight.*

Here is an example to demonstrate how easy sometimes it becomes to deal with certain SL systems with Liouville's reduction.

Example. The following Bessel Equation is a special case of SL equation (38), where $p(x) = \rho(x) = x,\ q = n^2/x$.

$$(xu')' + \left(k^2 x - \frac{n^2}{x}\right) u = 0.$$

Thus, Liouville's reduction (61) is $u = w/\sqrt{x}$ and $x = t$, which leads to the following equivalent DE

$$\frac{d^2w}{dx^2} + \left[k^2 - \frac{n^2 - \frac{1}{4}}{x^2}\right] w = 0, \qquad w = x^{1/2}u.$$

For $n = \frac{1}{2}$, the above becomes the trigonometric DE $w'' + k^2 w = 0$ which along with a periodic boundary condition, has $\{1,\ \cos kx,\ \sin kx\}$ (for $k = 1, 2, \cdots$) as a basis of solution. Since, we know that $J_{1/2}(0) = 0$, it follows that $J_{1/2}(x)$ is a constant multiple of $\sin x/\sqrt{x}$.

3.4 Modified Prüfer substitution

In the last section, we have seen that we can simplify the form of any regular SL equation through Liouville's transformation (61), and Corollary(3.3.2) further gaurantees that the SL system, in principle remains the same even after the transformation. Now, we intend to obtain asymptotic formulas for the n$^{\text{th}}$ eigenfunction $u_n(x)$, valid for large n, by applying a modification of the Prüfer substitution to the Liouville normal form of an SL system.

As mentioned, using (61) or Liouville's substituion in general, any regular SL system can be transformed into another regular SL system consisting of the equation

$$u'' + [\lambda - q(x)]u = u'' + Q(x)u = 0, \quad Q(x) = \lambda - q(x), \tag{66}$$

and separated boundary conditions of the same form

$$\alpha u(a) + \alpha' u'(a) = 0, \quad \beta u(b) + \beta' u'(b) = 0. \tag{67}$$

The constants $\alpha, \alpha', \beta, \beta'$ usually gets changed, but the relations $\alpha^2 + \alpha'^2 \neq 0$ and $\beta^2 + \beta'^2 \neq 0$ remains the same. Due to Corollary(3.3.2), the eigenvalues of the new system are the same as those of the original system, and the corresponding eigenfunctions are obtained for the Liouville normal form through the Liouville substitution. Therefore it suffices to consider system (66)-(67) in order to study the distribution of eigenvalues and magnitude of the eigenfunctions.

Here onwards, we will assume $Q(x) > 0$ in $[a, b]$, i.e., in (66) $\lambda > q(x)$ and also $Q \in C^1$. Now, we introduce $R(x, \lambda)$ and $\phi(x, \lambda)$, the new *Amplitutde variable function* and the new *Phase variable function*, respectively such that they are defined in terms of a given non-trivial solution $u(x, \lambda)$ of (66) by the equations

$$u = \frac{R}{\sqrt[4]{Q}} \cdot \sin\phi, \qquad u' = R\sqrt[4]{Q}\cos\phi. \tag{68}$$

We will call equations in (68) to be the nodified Prüfer substitution for the DE(66). Now, naturally, we shall derive a pair of DEs for R and ϕ, so that they together constitute to be equivalent to DE(66). Here we have

$$\cot\phi = \frac{1}{\sqrt{Q}}\frac{u'}{u}, \qquad R^2 = \sqrt{Q}u^2 + \frac{1}{\sqrt{Q}}u'^2. \tag{69}$$

Differentiating the first relation in (69) and using $u'' = -Qu$, we obtain

$$(\csc^2\phi)\phi' = \frac{Qu^2 + u'^2}{Q^{1/2}u^2} - \frac{1}{2}\frac{Q'}{Q^{3/2}}\frac{u'}{u}.$$

Now, using the second equation and then multiplying by $\sin^2\phi$, this will get simplfied to

$$\begin{aligned}(\csc^2\phi)\phi' &= \frac{R^2}{u^2} - \frac{1}{2}\frac{Q'}{Q}\cot\phi \\ \phi' &= Q^{1/2} - \frac{1}{4}\frac{Q'}{Q}\sin 2\phi.\end{aligned}$$

Moving ahead, we differentiate the second equation in (68) and obtain the identity

$$2RR' = 2Q^{-1/2}\left(Quu' + u'u''\right) + \left(\frac{Q'}{2Q}\right)\left(Q^{1/2}u^2 - Q^{-1/2}u'^2\right).$$

Above the first term must vanish as we have the relation $u'' = -Qu$, which leaves us with

$$\frac{R'}{R} = \frac{Q'}{4Q}\left(\sin^2\phi - \cos^2\phi\right) = \frac{-Q'}{4Q}\cos 2\phi.$$

Note that, equations in (69), and all the calculations succeeding them are only valid for $u \neq 0$. Whenever $u = 0$, we set $\tan\phi = \sqrt{Q}u/u'$ and proceed similarly.

Thus, in terms of λ and q, the modified Prüfer system is as follows :

$$\phi' = \sqrt{\lambda - q} + \frac{q'}{4(\lambda - q)}\sin 2\phi \tag{70}$$

$$\frac{R'}{R} = \frac{q'}{4(\lambda - q)}\cos 2\phi. \tag{71}$$

Clearly, to every non-trivial solution of (66) there is a corresponding solution of the modified Prüfer system (70)-(71), and conversely. Further, we also know that $R > 0$, unless R vanishes identically.

Before immediately presenting the result which says that Equations (70)-(71) determines the asymptotic behaviour of the solutions of (66) as $\lambda \to \infty$, we must get familair with some properties of the $\mathcal{O}$ (Big-oh) notation, particularly $\mathcal{O}(1)$.

1. We the basic algebraic properties

$$\mathcal{O}(1) + \mathcal{O}(1) = \mathcal{O}(1); \qquad \mathcal{O}(1) \cdot \mathcal{O}(1) = \mathcal{O}(1);$$

   and for any constant $k \in \mathbb{R}$, $\quad k \cdot \mathcal{O}(1) = \mathcal{O}(1)$.

2. If $\alpha,\ \beta \in \mathbb{R}$ such that $\alpha \leq \beta$, then we have

$$\frac{\mathcal{O}(1)}{\lambda^\alpha} + \frac{\mathcal{O}(1)}{\lambda^\beta} = \frac{\mathcal{O}(1)}{\lambda^\alpha}.$$

3. If $q(x)$ is any bounded function of x, then by Taylor's formula[1], we have

$$[\lambda - q(x)]^\alpha = \lambda^\alpha \left[1 - \frac{q(x)}{\lambda}\right]^\alpha = \lambda^\alpha - \alpha q(x)\lambda^{\alpha-1} + \mathcal{O}(1)\lambda^{\alpha-2}, \qquad \text{as } \lambda \to \infty.$$

Theorem 3.4.1. *Let $\phi(x,\lambda)$ and $R(x,\lambda)$ be solutions of the system (70) and (71), where $q(x) \in C^1$ is bounded. Then as $\lambda \to \infty$,*

$$\phi(x,\lambda) = \phi(a,\lambda) + \sqrt{\lambda}(x-a) + \frac{\mathcal{O}(1)}{\sqrt{\lambda}} \tag{72}$$

$$R(x,\lambda) = R(a,\lambda) + \frac{\mathcal{O}(1)}{\lambda}. \tag{73}$$

Proof. For all λ for which we have $|q(x)| < \lambda$ on $[a,b]$, we have $q/(\lambda - q) = \mathcal{O}(1)/\lambda$ and also using property 2. above we have the following,

$$\frac{q'}{\lambda - q} = \frac{q'}{\lambda}\Big(1 + \frac{\mathcal{O}(1)}{\lambda}\Big) = \frac{q'}{\lambda} + \frac{\mathcal{O}(1)}{\lambda^2},$$

$$\sqrt{\lambda - q} = \sqrt{\lambda}\Big(1 - \frac{q}{\lambda}\Big)^{1/2} = \sqrt{\lambda} - \frac{q}{2\sqrt{\lambda}} + \frac{\mathcal{O}(1)}{\lambda^{3/2}}.$$

Observe that $\phi_1(x,\lambda) = \phi(a,\lambda) + \sqrt{\lambda}(x-a)$ and $R_1(x,\lambda) \equiv R(a)$ are two particular solutions of the following DEs,

$$\phi' = \sqrt{\lambda} \quad \text{and} \quad (\log R)' = 0.$$

correspondingly. Now we compare the solutions of DE (70) and (71) with the above mentioned solutions ϕ_1 and R_1, one comaprison at a time. For the first comparison, we select $\epsilon = \mathcal{O}(1)/\sqrt{\lambda}$, and replace $\mathbf{x}$ and $\mathbf{y}$ with the functions $\phi(x,\lambda)$ and $\phi_1(x,\lambda)$, respectively. As the initial conditions are same, i.e., $\phi(a,\lambda) = \phi(a,\lambda)$, we get $|\phi(x,\lambda) - \phi_1(x,\lambda)| \leq \mathcal{O}(1)/\sqrt{\lambda}$, further using properties of $\mathcal{O}(1)$, equation (72) follows.

Next, we go for the second comparison, which needs a slight manipulation. Note that, using Taylor's formula we already have the identity $e^{\mathcal{O}(1)/\lambda} = 1 + \mathcal{O}(1)/\lambda$. In the Comparison Theorem(I.III.II), for $\epsilon = \mathcal{O}(1)/\lambda$, we get the inequality $|\log R(x,\lambda) - \log R_1(x,\lambda)| \leq \epsilon$.

The calculations proceeds as follows :

$$
\begin{aligned}
\implies \qquad & \log\left[\frac{R(x,\lambda)}{R_1(x,\lambda)}\right] = \epsilon \\
\implies \qquad & e^{\log\left[\frac{R(x,\lambda)}{R_1(x,\lambda)}\right]} = e^{\epsilon} = e^{\mathcal{O}(1)/\lambda} \\
\implies \qquad & \frac{R(x,\lambda)}{R_1(x,\lambda)} = 1 + \frac{\mathcal{O}(1)}{\lambda} \\
\implies \qquad & R(x,\lambda) = R(a,\lambda) + R(a,\lambda)\cdot\frac{\mathcal{O}(1)}{\lambda} \\
\implies \qquad & R(x,\lambda) = R(a,\lambda) + \frac{\mathcal{O}(1)}{\lambda}.
\end{aligned}
$$

Hence, we deduce equation (73). This completes the proof. □

The key takeaway from Theorem(3.4.1) mentioned above is that, intuitively, for large enough λ, the *modified Phase variable function* $\phi(x,\lambda)$ approximately behaves as a linear function of $\sqrt{\lambda}$ and the *modified Amplitutde variable function* $R(x,\lambda)$ approximately behaves as a constant.

3.5 Distribution of eigenvalues

Both in the current Section(3.5) and in the next section, we will treat in detail, the regular SL systems satisfying separated endpoint condition (39) so that $\alpha'\beta' \neq 0$. Before straight-away presenting the context, let us review some examples.

Examples. Here are some examples concerning the *trigonometric DE* $u'' + \lambda u = 0$ in an interval $[a,b]$.

1. The *trigonometric DE* in an interval $[a,b]$ along with the boundary condition $u(a) = u(b) = 0$ has its n$^{\text{th}}$ eigenfunction $u_n(x) = \sin[n\pi(x-a)/(b-a)]$, corresponding to its n$^{\text{th}}$ eigenvalue $\lambda_n = n^2\pi^2/(b-a)^2$, for n running over $\mathbb{N}$.

2. Again for the same mentioned DE, but with endpoint condition $u(a) = u'(b) = 0$, we get $u_n(x) = \sin\sqrt{\lambda_n}(x-a)$, where $\lambda_n = (n+\frac{1}{2})^2\pi^2/(b-a)^2$, such that n runs over $\mathbb{N}$.

3. Also, when $u'(a) = u'(b) = 0$, we get its (n+1)$^{\text{th}}$ eigenfunction as $u_{n+1}(x) = \cos\sqrt{\lambda_n}(x-a)$, where $\lambda_n = n^2\pi^2/(b-a)^2$, for n running over $\mathbb{N}_0$.

In this section, we shall see that the cases of the trigonometric DE is just typical, and hence shall present result showing that the asymptotic distribution of eigenvalues of all regular SL systems is the same. In particular, we will treat in detail the case of separated endpoint conditions (39), further assuming $\alpha'\beta' \neq 0$. We can safely consider, for our convenience, the given SL system reduced in its Liouville normal form (66)-(67), as anyways this does not change the eigenvalues or the condition $\alpha'\beta' \neq 0$.

Our aim would be to show that $\sqrt{\lambda_n} = [n\pi/(b-a)] + \mathcal{O}(1)/(n)$ for $n = 0, 1, 2, \cdots$. Basically, unless $\alpha' = \beta' = 0$ in (67), the asymptotic behavior of the eigenvalues and eigenfunctions is similar to that of $u'' + \lambda u = 0$, with the endpoint conditions $\alpha = \beta = 0$. Let $A = \alpha/\alpha'$ and $B = -\beta/\beta'$. By the assumption $\alpha'\beta' \neq 0$, we know A and B to be finite. We choose $\phi(x, \lambda)$, a solution of (66) satisfying the initial condtition

$$\cot \phi(a, \lambda) = \frac{A}{\sqrt{\lambda - q(a)}}, \qquad 0 \leq \phi(a, \lambda) < \pi. \tag{74}$$

According to (68), the non-trivial solution $u(x)$ corresponding to $\phi(x, \lambda)$ will be an eigenfunction iff

$$\cot \phi(b, \lambda) = \frac{B}{\sqrt{\lambda - q(b)}}, \qquad \phi(b, \lambda) = \delta + n\pi,\ \delta \in (0, \pi]. \tag{75}$$

By using the expansion of $\operatorname{arccot} x$, around $x = 0$, we simplify relation (74). For $\lambda \to \infty$, we have

$$\phi(a, \lambda) = \frac{\pi}{2} - \frac{A}{\sqrt{\lambda}} + \frac{\mathcal{O}(1)}{\lambda^{3/2}}. \tag{76}$$

By virtue of Theorem(3.2.1), we can mark the above conclusion as, for large value of $n \in \mathbb{N}_0$, we have

$$\boxed{\phi(a, \lambda_n) = \frac{\pi}{2} - \frac{A}{\sqrt{\lambda_n}} + \frac{\mathcal{O}(1)}{\lambda_n^{3/2}}.} \tag{77}$$

Similarly, we simply condition (75), to get that that for large value of $n \in \mathbb{N}_0$,

$$\boxed{\phi(b, \lambda_n) = n\pi + \frac{\pi}{2} - \frac{B}{\sqrt{\lambda_n}} + \frac{\mathcal{O}(1)}{\lambda_n^{3/2}}.} \tag{78}$$

Also note that, directly invoking equation (72), we get that for large value of $n \in \mathbb{N}_0$,

$$\begin{aligned} \phi(x, \lambda_n) &= \frac{\pi}{2} + \sqrt{\lambda_n}(x-a) + \frac{\mathcal{O}(1)}{\sqrt{\lambda_n}} \\ \implies \phi(b, \lambda_n) &= \phi(a, \lambda_n) + \sqrt{\lambda_n}(b-a) + \frac{\mathcal{O}(1)}{\sqrt{\lambda_n}} \\ \implies \phi(b, \lambda_n) - \phi(a, \lambda_n) &= \sqrt{\lambda_n}(b-a) + \frac{\mathcal{O}(1)}{\sqrt{\lambda_n}}. \end{aligned} \tag{79}$$

We finally combining equations (77), (78) and (79) we get that for large value of $n \in \mathbb{N}_0$,

$$\boxed{\phi(b, \lambda_n) - \phi(a, \lambda_n) = n\pi + \frac{\mathcal{O}(1)}{\sqrt{\lambda_n}} = \sqrt{\lambda_n}(b-a) + \frac{\mathcal{O}(1)}{\sqrt{\lambda_n}}.} \tag{80}$$

For $n \to \infty$ as $\lambda_n \to \infty$, we obtain $\lim_{n\to\infty} n\pi/\sqrt{\lambda_n} = (b-a)$, or $\sqrt{\lambda_n} = K_n n$, such that K_n goes to $\pi/(b-a)$. Substituting these in equation (80), we obtain

$$\boxed{\sqrt{\lambda_n} = \frac{n\pi}{b-a} + \frac{\mathcal{O}(1)}{\sqrt{\lambda_n}} = \frac{n\pi}{b-a} + \frac{\mathcal{O}(1)}{n}.}$$

Hence, we present all these manipulative calculations above in the form of the following theorem and its corollary.

Theorem 3.5.1. *For the regular SL system (66)-(67), let $\alpha'\beta' \neq 0$. Then the eigenvalues λ_n are given, as $n \to \infty$, by the following asymptotic formula*

$$\sqrt{\lambda_n} = \frac{n\pi}{b-a} + \frac{\mathcal{O}(1)}{n}.$$

Here, $\mathcal{O}(1)$ denotes a function uniformly bounded for all integers $n \geq 0$.

Corollary 3.5.2. *If $\{\lambda_n\}$ is the sequence of nonzero eigenvalues of a regular SL system, then $\Sigma_{n=0}^{\infty}\lambda^{-2} < \infty$.*

3.6 Normalised eigenfunctions

Definition 3.6.1. ***Normalized square-integrable functions*** *A square-integrable function u on an interval (a,b) is normalized relative to a weight function ρ when*

$$\int_a^b u^2(x)\rho(x)dx = 1.$$

Note that, in the case of the eigenfunctions of (66), $\rho(x) \equiv 1$. For this section, our aim is to show that the normalized eigenfunctions of (66) and (67) behave approximately like cosine functions, with the additional constraint $\alpha'\beta' \neq 0$.

Due to the first relation in (68), for an eigenfunction $u_n(x)$, with eigenvalue λ_n, we have

$$\boxed{u_n(x) = \frac{R(x,\lambda_n)}{\sqrt[4]{\lambda_n - q(x)}}\sin\phi(x,\lambda_n),} \qquad a \leq x \leq b. \tag{81}$$

For $\lambda \to \infty$, we use (72), to get

$$\begin{aligned}
\frac{d\phi}{dx} &= \sqrt{\lambda} + \frac{\mathcal{O}(1)}{\sqrt{\lambda}} \\
\frac{dx}{d\phi} &= \frac{1}{\sqrt{\lambda}(1+\frac{\mathcal{O}(1)}{\lambda})} \\
&= \frac{1}{\sqrt{\lambda}} \cdot \sum_{k=0}^{\infty}(-1)^k\Big(\frac{\mathcal{O}(1)}{\lambda}\Big)^k \qquad \text{[using convergence of geometric series]} \\
&= \frac{1}{\sqrt{\lambda}}\Big(1 - \frac{\mathcal{O}(1)}{\lambda}\Big) \\
&= \frac{1}{\sqrt{\lambda}} + \frac{\mathcal{O}(1)}{\lambda^{3/2}}. \qquad \text{[using property of } \mathcal{O}(1)\text{]} \\
\therefore\ \frac{dx}{d\phi} &= \frac{1}{\sqrt{\lambda}} + \frac{\mathcal{O}(1)}{\lambda^{3/2}}.
\end{aligned} \tag{82}$$

We know consider the following integral for $\lambda \to \infty$,

$$\begin{aligned}
\int_a^b \sin^2 \phi(x,\lambda)dx &= \int_{\phi(a,\lambda)}^{\phi(b,\lambda)} \sin^2 \phi \frac{dx}{d\phi} d\phi \\
&= (\lambda^{-1/2} + \mathcal{O}(1)\lambda^{-3/2}) \int_{\phi(a,\lambda)}^{\phi(b,\lambda)} \sin^2 d\phi \qquad \text{[using (82)]} \\
&= (\lambda^{-1/2} + \mathcal{O}(1)\lambda^{-3/2}) \Big[\frac{\phi}{2} - \frac{\sin 2\phi}{4}\Big]_{\phi(a,\lambda)}^{\phi(b,\lambda)} \\
&= (\lambda^{-1/2} + \mathcal{O}(1)\lambda^{-3/2}) \cdot \big[\frac{\lambda^{1/2}(b-a)}{2} + \mathcal{O}(1)\big] \qquad \text{[using (80)]} \\
&= \frac{(b-a)}{2} + \frac{\mathcal{O}(1)}{\lambda^{1/2}}.
\end{aligned}$$

We must present the above calculations in the form of the following lemma so that we can use it later.

Lemma 3.6.1. *Let $\phi(x,\lambda)$ be as in the proof of Theorem(3.5.1). Then as $\lambda \to \infty$,*

$$\int_a^b \sin^2 \phi(x,\lambda)dx = \frac{(b-a)}{2} + \frac{\mathcal{O}(1)}{\lambda^{1/2}}.$$

A second step towards our primary aim of this section is the following lemma.

Lemma 3.6.2. *Let $u(x,\lambda)$ be a solution of (66). Then as $\lambda \to \infty$,*

$$\Big(\int_a^b u^2(x)dx\Big)^{1/2} = R(a,\lambda) \cdot \lambda^{-1/4} \cdot \sqrt{\frac{b-a}{2}} \Big(1 + \frac{\mathcal{O}(1)}{\lambda^{1/2}}\Big) + \frac{\mathcal{O}(1)}{\lambda^{5/4}}.$$

Proof. Using the first relation in (68) and then expanding $R(x,\lambda)$ as in equation (73), we get

$$\int_a^b u^2(x)dx = \Big[R(a,\lambda) + \frac{\mathcal{O}(1)}{\lambda}\Big]^2 \int_a^b [\lambda - q(x)]^{-1/2} \sin^2 \phi dx.$$

We use property(3) of $\mathcal{O}(1)$ mentioned above to deduce that $[\lambda - q]^{-1/2} = \lambda^{-1/2} + \mathcal{O}(1)\lambda^{-3/2}$. Further using Lemma(3.6.1) to simply the above integral, we get

$$\begin{aligned}
\int_a^b u^2(x)dx &= \Big[R(a,\lambda) + \frac{\mathcal{O}(1)}{\lambda}\Big]^2 (\lambda^{-1/2} + \mathcal{O}(1)\lambda^{-3/2}) \Big(\frac{b-a}{2} + \mathcal{O}(1)\lambda^{-1/2}\Big) \\
&= \Big[R(a,\lambda) + \frac{\mathcal{O}(1)}{\lambda}\Big]^2 \Big(\frac{b-a}{2\lambda^{1/2}} + \frac{\mathcal{O}(1)}{\lambda}\Big) \\
\Big(\int_a^b u^2(x)dx\Big)^{1/2} &= \Big(R(a,\lambda) + \frac{\mathcal{O}(1)}{\lambda}\Big) \Big(\frac{b-a}{2\lambda^{1/2}} + \frac{\mathcal{O}(1)}{\lambda}\Big)^{1/2} \\
&= \frac{R(a,\lambda)}{\lambda^{1/4}} \sqrt{\frac{b-a}{2}} \Big(1 + \frac{\mathcal{O}(1)}{\lambda^{1/2}}\Big) + \frac{\mathcal{O}(1)}{\lambda^{5/4}}.
\end{aligned}$$

This completes the proof. □

Corollary 3.6.3. *If, in Lemma(3.6.2),* $\int_a^b u^2(x,\lambda)dx = 1$*, then*

$$R(a,\lambda) = \sqrt{\frac{2}{b-a}}\lambda^{1/4}\left[1 + \mathcal{O}(1)\lambda^{-1/2}\right].$$

Proof. From Lemma(3.6.2), for normalized solutions, we get :

$$1 - \frac{\mathcal{O}(1)}{\lambda^{5/4}} = \frac{R(a,\lambda)}{\lambda^{1/4}} \cdot \sqrt{\frac{b-a}{2}}\left(1 + \frac{\mathcal{O}(1)}{\lambda^{1/2}}\right).$$

Here, by solving for using appropriate property of $\mathcal{O}(1)$, we get the relation we wanted to deduce. This completes the proof. □

Lemma 3.6.4. *Let* λ_n *be the* n^{th} *eigenvalue of the SL system (66)-(67). Then, as* $n \to \infty$ *we have the following unless* $\alpha'\beta' = 0$*,*

$$\sin\phi\,(x,\lambda_n) = \cos\frac{n\pi(x-a)}{b-a} + \mathcal{O}(1)\lambda_n^{-1/2}.$$

Proof. By Theorem(3.4.1), (72) we have

$$\phi(x,\lambda_n) = \phi(a,\lambda_n) + \sqrt{\lambda_n}(x-a) + \frac{\mathcal{O}(1)}{\sqrt{\lambda_n}}.$$

Moreover from (77), we borrow $\phi(a,\lambda_n) = \pi/2 + \mathcal{O}(1)/\sqrt{\lambda_n}$. Thus, substituting this in the previous formula and using the identity $\sin(A + \mathcal{O}(1)/x^n) = \sin A + \mathcal{O}(1)/x^n$, for positive n, we get :

$$\begin{aligned}\sin\phi\,(x,\lambda_n) &= \sin\left[\sqrt{\lambda_n}(x-a) + \pi/2\right] + \mathcal{O}(1)/\sqrt{\lambda_n}\\ \sin\phi\,(x,\lambda_n) &= \cos\left[\sqrt{\lambda_n}(x-a)\right] + \mathcal{O}(1)/\sqrt{\lambda_n}.\end{aligned} \tag{83}$$

We now apply Theorem(3.5.1), from the previous section on the argument $\sqrt{\lambda_n}(x-a)$. Also, we borrow $\mathcal{O}(1)/n = \mathcal{O}(1)/\sqrt{\lambda_n}$ from the last section to get

$$\cos\left[\sqrt{\lambda_n}(x-a)\right] - \cos\left[\frac{n\pi(x-a)}{(b-a)}\right] = \mathcal{O}(1)n^{-1} = \mathcal{O}(1)\lambda_n^{-1/2}$$

Substituting this appropriately in (83), we get what we needed.

$$\sin\phi\,(x,\lambda_n) = \cos\frac{n\pi(x-a)}{b-a} + \mathcal{O}(1)\lambda_n^{-1/2}. \tag{84}$$

This completes the proof. □

Now let us focus back on the relation (81). $1/\sqrt[4]{\lambda_n - q(x)}$ can be replaced by the approximation $(\lambda - q)^{-1/4} = \lambda^{-1/4} + \mathcal{O}(1)\lambda^{-5/4}$. The factor $R(x,\lambda_n)$ is again approximated by Corollary(3.6.3), and lastly $\sin\phi(x,\lambda)$ is estimated due to Lemma(3.6.4). Thus, these substitutions allows us to obtain

$$\boxed{u_n(x) = \sqrt{\frac{2}{b-a}}\cos\left[\frac{n\pi(x-a)}{b-a}\right] + \mathcal{O}(1)\lambda_n^{-1/2}.}$$

As $\lambda^{-1/2} = \mathcal{O}(1)n^{-1}$, we complete the proof and hence, present the theorem which imples the primary aim of this section.

Theorem 3.6.5. *Let $\{u_n\}$, for n running over $\mathbb{N}_0$, be the sequence of normalised eigenfunctions of the regular SL system (66)-(67), with $\alpha'\beta' \neq 0$. Then*

$$u_n(x) = \sqrt{\frac{2}{b-a}}\cos\left[\frac{n\pi(x-a)}{b-a}\right] + \frac{\mathcal{O}(1)}{n}. \tag{85}$$

Change in notation. Please note that, from the upcoming section, the symbol ϕ will no longer be used to denote the usual Phase variable function in context of Prüfer or Modified Prüfer substitution. The entity which it will denote will be mentioned appropriately.

3.7 Orthogonal expansions

Let us assume $\phi_1(x),\ \phi_2(x),\ \phi_3(x),\cdots$ to be any bounded, square-integrable functions on an interval $\tilde{I} : a < x < b$, orthogonal with respect to a positive weight function $\rho(x)$, so that

$$\int_{\tilde{I}} \phi_h(x)\phi_k(x)\rho(x)dx = 0, \qquad \text{if } h \neq k.$$

Suppose that a given function $f(x)$ can be expressed as the limit of a uniformly convergent series of multiples of the functions $\phi_k(x)$, so that we have

$$f(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + \cdots = \sum_{h=1}^{\infty} c_h\phi_h(x) \tag{86}$$

As the assumption of uniform convergence allows us to have term-by-term integration of an infinite series, we multiply both sides of (86) by $\phi_k(x)\rho(x)$, and then integrate term-by-term over the interval. By virtue of the above mentioned orthogonality relation, we get the following equation

$$\int_{\tilde{I}} f(x)\phi_k(x)\rho(x)dx = \sum_{h=1}^{\infty}\int_{\tilde{I}} c_h\phi_h(x)\phi_k(x)\rho(x)dx = c_k\int_{\tilde{I}}\phi_k^2(x)\rho(x)dx$$

Hence, the coefficients c_h in (86) must satisfy and can be computed thorugh, the following relation

$$c_h = \left\{\int_{\tilde{I}} f(x)\phi_h(x)\rho(x)dx\right\} \Big/ \left\{\int_{\tilde{I}}\phi_h^2(x)\rho(x)dx\right\}. \tag{87}$$

Here, one can observe the similarity of the above deduction with the one done in Chapter(1) for deducing the Fourier coefficients. In fact, when $\phi_k(x)$ are the trigonometric sinusoid functions, we precisely obtain, as a special case, the coefficients of the Fourier series with $\rho(x) = 1$. Now, we can summarize the above discussion as the following result.

Theorem 3.7.1. *If a function $f(x)$ is the limit $f(x) = \Sigma c_k\phi_k(x)$ of a uniformly convergent series of constant multiples of bounded square-integrable functions $\phi_k(x)$ that are orthogonal with respect to a weight function $\rho(x)$, the coefficients c_h of the series are given by*

$$c_h = \frac{\int_{\tilde{I}} f(x)\phi_h(x)\rho(x)dx}{\int_{\tilde{I}} \phi_h^2(x)\rho(x)dx}.$$

Note that, the preceding conclusion, i.e., Theorem(3.7.1) holds provided that one can integrate the series $\Sigma c_h\phi_h(x)\phi_k(x)\rho(x)$ term-by-term on the interval $\tilde{I}$. Well, this holds much more generally than for uniform convergence, for example, for mean-square convergence, which we will discuss in the upcoming Section(3.5).

3.8 Mean-square approximation : L^2_ρ - convergence

Till now, we have put only uniformly convergent series under our consideration, since these allows us to perform term-by-term integration. The notion of convergence most appropriate for orthogonal expansions is, however, not uniform convergence but mean-square convergence, which we define as follows.

Definition 3.8.1. *Let f and the terms of the sequence $\{f_n\}_{n\in\mathbb{N}}$ be square-integrable real functions. The sequence $\{f_n\}$ is said to converge to f in the mean square on the interval $\tilde{I}$, with respect to the positive weight function $\rho(x)$, when*

$$\int_{\tilde{I}} [f_n(x) - f(x)]^2\rho(x)dx \to 0, \qquad \text{as } n \to \infty. \tag{88}$$

Now, suppose that $\phi_1,\ \phi_2,\ \phi_3, \cdots$ form an infinite sequence of square-integrable functions on the interval $\tilde{I}$, orthogonal with respect to the weight function $\rho(x)$, and let $f_n(x) = \gamma_1\phi_1(x) + \cdots + \gamma_n\phi_n(x)$ to be the n$^{\text{th}}$ partial sum of the series $\Sigma_{k=1}^{\infty}\gamma_k\phi_k(x)$. To make the partial sums f_n, converge in the mean-square to f as rapidly as possible, we choose the coefficients γ_n, so as to minimize the following expression :

$$E = E(\gamma_1, \ldots, \gamma_n) = \int_{\tilde{I}} \left[f(x) - \sum_{k=1}^{n} \gamma_k\phi_k(x) \right]^2 \rho(x)dx. \tag{89}$$

We now expand (89) and use the orthogonality relation mentioned above, we get the following expression for the function E in variables $\gamma_1,\ \gamma_2, \cdots,\ \gamma_n$

$$E = \int_{\tilde{I}} f^2\rho\ dx - 2\sum_{k=1}^{n}\gamma_k \int_{\tilde{I}} f\phi_k\rho\ dx + \sum_{k=1}^{n}\gamma_k^2 \int_{\tilde{I}} {\phi_k}^2\rho\ dx.$$

We purposefully consider the numbers $\gamma_1,\ \gamma_2, \cdots,\ \gamma_n$ so that they minimize E. As E is differentiable in each of its variables, the minimum of E can be attained only by setting every $\partial E/\partial y = 0$. Basically, a necessary condition for a minimum is that the γ_k satisfy the following equation

$$-2\int_{\tilde{I}} f\phi_k\rho dx + 2\gamma_k\int_{\tilde{I}}\phi_k^2\rho\ dx = 0$$

$$\gamma_k = \frac{\int_{\tilde{I}} f(x)\phi_k(x)\rho(x)\ dx}{\int_{\tilde{I}}\phi_k^2(x)\rho(x)\ dx}.$$

It is worth noticing that, the expression for the γ_k minimizing the *mean-square error* E is same as the coefficients c_k in (87). Therefore, the choice $\gamma_k = c_k$ as in (87) does indeed give a minimum for E. A simple calculation, completing the square, gives us the expression of E as follows.

$$\begin{aligned} E &= \int_{\tilde{I}}\left[f - \sum\gamma_k\phi_k\right]^2\rho\ dx \\ &= \int_{\tilde{I}} f^2\rho\ dx + \sum_{k=1}^{n}\left[-c_k^2 + (\gamma_k - c_k)^2\right]\int_{\tilde{I}}\phi_k^2\rho\ dx. \end{aligned}$$

Here, in the above expression it is evident that the minimum of E is attained iff $\gamma_k = c_k$. These discussion completes the proof and enables us to present following result, for any interval $\tilde{I}$.

Theorem 3.8.2. *Let $\{\phi_k(x)\}$ be a sequence of orthogonal square-integrable functions, and let f be square-integrable. Then, among all possible choices of $\gamma_1,\ \gamma_2, \cdots,\ \gamma_n$ the integral in (89) is minimised by selecting $\gamma_k = c_k$, where c_k is defined as*

$$c_k = \frac{\int_{\tilde{I}} f(x)\phi_k(x)\rho(x)\ dx}{\int_{\tilde{I}}\phi_k^2(x)\rho(x)\ dx}.$$

The partial sum $c_1\phi_1 + \cdots + c_n\phi_n$ in Theorem(3.7.1) is therefore, the *best mean-square approximation* to $f(x)$ among all possible sums $\gamma_1\phi_1 + \cdots + \gamma_n\phi_n$, and it is often called the *least square approximation* to $f(x)$ because it minimizes the mean square difference E in (89).

3.8.1 Orthonormal functions

The preceding formulas like (87) become much simpler if the orthogonal functions ϕ_k are *orthonormal*, that is apart from being orthogonal they also satisfy $\int_{\tilde{I}}\phi_k^2\rho(x)\ dx = 1$. We can easily construct, from any sequence $\{\phi_k\}$ of orthogonal functions, an orthonormal sequence $\{\psi_k\}$, by setting $\psi_k = \phi_k \Big/ \sqrt{\int_{\tilde{I}}\phi_k^2\rho(x)\ dx}$. Note that, through all the above expressions E is non-negative, so by substituting the relation $\int_{\tilde{I}}\phi_k^2\rho(x)\ dx = 1$, we get the following result.

Corollary 3.8.3. *Let $\Sigma_1^n c_k\phi_k$ be the least-square approximation to f by a linear combination of orthonormal functions ϕ_k. Then*

$$\sum_{k=1}^{n} c_k^2 \leq \int_{\tilde{I}} f^2(x)\rho(x)dx. \tag{90}$$

For the right-hand-side member of (90) to be finite, it is necessary that the function $f^2(x)\rho(x)$ be integrable, that is, that f be square-integrable with respect to the weight function $\rho(x)$. When this is the case, then due to Chauchy-Schwartz Inequality, the integrals in (87) are also well-defined. As the right-hand-side member of (90) is independent of n, we can let n tend to ∞, which will give us

$$\sum_{k=1}^{\infty} c_k^2 \leq \int_I f^2(x)\rho(x)dx < +\infty. \tag{91}$$

The inequality (91) is a celebrated result in mathematical analysis in general, and is popularly called as the *Bessel's Inequality.*

3.8.2 Completeness

Now let us proceed ahead in the direction which matters to us the most in the current context. The most important question about a sequence of continuous functions ϕ_k $(k = 1, 2, 3, \cdots)$, orthogonal and square-integrable with respect to a weight function $\rho(x)$, is : *Can every square-integrable function f be expanded into an infinite series $f = \Sigma_1^\infty c_k\phi_k$ of ϕ_k? When this is possible for every continuous f, the sequence of orthogonal functions ϕ_k is said to be* ***complete***.†

Using the fundamental form of the error in mean-square approximation, mentioned above, we can reformulate the definition of completeness as follows. In order to the following to happen

$$\lim_{n\to\infty} \int_{\tilde{I}} \left[f(x) - \sum_{k=1}^{n} \gamma_k \phi_k \right]^2 \rho(x)dx = 0,$$

it is necessary and sufficient that we have

$$\lim_{n\to\infty} \left\{ \left[\int_{\tilde{I}} f^2 \rho dx - \sum_{k=1}^{n} c_k^2 \int_{\tilde{I}} \phi_k^2 \rho dx \right] + \sum_{k=1}^{n} (\gamma_k - c_k)^2 \int_{\tilde{I}} \phi_k^2 \rho dx \right\} = 0.$$

By Bessel's Inequality (91), we know that the term in square-braces in the above expression is non-negative. Also, since $\int \phi_k^2 \rho \ dx > 0$ for any non-trivial ϕ_k, the limit is zero iff $\gamma_k = c_k \ \forall k$, and equality holds in Bessel's inequality (91). This proves the following result.

We present Theorem(3.8.4) and using the definition of orthonormal functions, its Corollary(3.8.6) as follows.

† Note that, Here and below, the equation $f = \Sigma_1^\infty c_k\phi_k$ is to be interpreted in the sense of mean-square convergence, that is, the partial sums $\Sigma_1^\infty c_k\phi_k$ converge in the mean square to the function f with respect to ρ. Further, if every continuous function can be expanded into a series $\Sigma_1^\infty c_k\phi_k$, then many discontinuous functions also have such an expansion, convergent in the mean square. The class of all such functions is that of all Lebesgue square-integrable functions.

Theorem 3.8.4. *A sequence $\{\phi_k\}$ for functions $\phi_k(x)$, orthogonal and square-integrable with positive weight function $\rho(x)$ on an interval $\tilde{I}$, is complete iff for all continuous square-integrable functions f, we have*

$$\int_{\tilde{I}} f^2(x)\rho(x)dx = \sum_{k=1}^{\infty} \left\{ \left[\int_{\tilde{I}} f(x)\phi_k(x)\rho(x)dx \right]^2 \Big/ \int_{\tilde{I}} \phi_k^2(x)\rho(x)dx \right\}.$$

Corollary 3.8.6. *If the functions $\phi_k(x)$ are orthonormal, then a necessary and sufficient condition for completeness is, that for all continuous square-integrable functions f, the following Parseval's Identity holds, i.e.,*

$$\int_{\tilde{I}} f^2(x)\rho(x)dx = \sum_{k=1}^{\infty} \left[\int_{\tilde{I}} f(x)\phi_k(x)\rho(x)dx \right]^2.$$

Observe that by simply using the above Corollary(3.8.6), one can deduce the completeness of $\{1,\ \cos kx,\ \sin kx \mid k \text{ running over } \mathbb{N}\}$ (i.e. L^2 convergence of Fourier series). In Chapter(1), we have seen the validity of the Parseval's Identity for a Fourier series.

We can now conclude this section with the following theorem, which provides a criterion for completeness of a sequence of orthogonal functions, which relates the notion of completeness to that of approximation in the sense of mean-square convergence or L^2_ρ-convergence. Note that for the following theorem, one direction of its proof follows clearly from the discussions done in this section, while the other direction is trivial.

Theorem 3.8.7. *Let $\{\phi_k\}_{k\in\mathbb{N}}$ be any sequence of orthogonal square-integrable functions on an interval $\tilde{I}$, relative to a weight function $\rho(x) > 0$. The sequence is complete iff every continuous square-integrable function can be approximated arbitrarily closely in the mean square by a linear combination of the ϕ_ks.*

3.9 Completness of eigenfunctions - I

We, finally intend to present the completeness of the eigenfunctions of a regular Sturm-Liouville system in this section, which we will see, to be a consequence of the asymptotic formulas discussed throughout Section(3.5) and Section(3.6). Further, the completeness also depends on a geometric property of sets of orthonormal vectors in Euclidean vector space[2]. This property is stated in the following crucial theorem, often credited to N. Bary.

Theorem 3.9.1. *Let $\{\phi_n\}$ be any complete sequence of orthonormal vectors in a Euclidean vector space E, and let $\{\psi_n\}$ be any sequence of orthonormal vectors in E, such that it satisfies the inequality*

$$\sum_{n=1}^{\infty} \|\psi_n - \phi_n\|^2 < +\infty.$$

Then the vectors ψ_n are complete in E.

While the proof for Theorem(3.9.1) is crucial for accepting of its validity, but due to limitations in focus, the detailed exposition of its proof will be skipped here. The proof, however, is readily available in the literature [2], and its validity has been verified in various contexts. Although, intuitively it is clear, since it asserts that completeness is preserved in passing from a set of orthonormal vectors ϕ_n to any nearby system.

Assuming Theorem(3.9.1), we will try to establish the completeness of the eigenfunctions of a regular SL system.

Proposition 3.9.2. *The functions* $\cos[k\pi(x-a)/(b-a)]$*, for* $k = 0, 1, 2, \ldots$*, form a complete orthogonal sequence in* $L^2[a, b]$*.*

Proof. Note that, as $\{1, \cos kx, \sin kx \mid k \text{ running over } \mathbb{N}\}$ is complete in $L^2[a, b]$, through changing variables we can similarly establish the validity of Parseval's equality for functions $\cos[k\pi(x-a)/(b-a)]$, for k running over $\mathbb{N}_0$. Hence, we get what we wanted to proof. □

From Theorem(3.6.5) stated in Section(3.6), we consider the asymptotic formula (85),

$$u_n(x) = \sqrt{\frac{2}{b-a}} \cos\left[\frac{n\pi(x-a)}{b-a}\right] + \frac{\mathcal{O}(1)}{n}.$$

If $u_n(x)$ is the n^{th} normalized eigenfunction of a regular SL system in Liouville normal form, and if $\phi_n(x) = \sqrt{2/(b-a)} \cos[(n\pi(x-a))/(b-a)]$, then we have $|u_n(x) - \phi_n(x)| = \mathcal{O}(1)/n$. It follows that,

$$\|u_n - \phi_n\|^2 = \int_{\tilde{I}} [u_n(x) - \phi_n(x)]^2 \, dx = \frac{\mathcal{O}(1)}{n^2}.$$

As the infinite series $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \cdots$ converges (to $\pi^2/6$), we can present the following lemma out of this discussion.

Lemma 3.9.3. *Let* $u_n(x)$ *be the* n^{th} *normalized eigenfunction of any regular SL system in Lioville normal form, along with the constraint* $\alpha'\beta' \neq 0$*, and let*

$$u_n(x) = \sqrt{\frac{2}{b-a}} \cos\left[\frac{n\pi(x-a)}{b-a}\right].$$

Then the ϕ_n *are an orthogonal sequence, and also*

$$\sum_{n=1}^{\infty} \|u_n - \phi_n\|^2 < +\infty.$$

Now from a combined application of Proposition(3.9.2), Lemma(3.9.3) and Bary's Theorem(3.9.1), it follows that the eigenfunctions of any regular SL system in Liouville normal form with constraint $\alpha'\beta' \neq 0$ are a complete set of orthonormal functions in the appropriate L^2 space.

As shown in Section(3.3), when applied to normalized eigenfunctions, the transformation to Liouville normal form carries the inner product $< \phi, \psi >= \int_{\tilde{I}} \phi(x)\psi(x)\rho(x) \, dx$ into the inner product $< u, v >= \int_0^c u(x)v(x)\, dx$. Therefore, the change of variable that leads to a Liouville normal form carries complete orthonormal sequences (relative to weight function $\rho(x)$) again to complete orthonormal sequences. Therefore, we can conclude that, *the eigenfunctions of regular SL systems, which are not in Liouville normal form are also complete.* As very similar arguments cover the case with constraints $\alpha'\beta' = 0$, our goal in the larger persepective is hence established in the form of the following theorem.

Theorem 3.9.4. *The eigenfunctions of any regular SL system are complete in the space of all square-integrable functions, on the interval* $[a, b]$*, relative to the weight function* $\rho(x)$*.*

3.10 Completness of eigenfunctions - II

It is time, that we shall see the completeness of eigenfunctions of some of the singular SL systems we have previously seen in Section(2.3). Particularly, we will consider SL systems on a finite interval whose eigenfunctions are polynomials, such as the Legendre polynomials. We can use any positive weight function $\rho(x)$ on an interval (a, b) such that the integral $\int_a^b x^n \rho(x)dx$ is convergent for all $n \geq 0$. The plan is to construct an infinite sequence of polynomials $P_0(x), P_1(x), \cdots$, with $P_n(x)$ of degree n, which are orthogonal on (a, b) with respect to this weight function, so that we have

$$\int_a^b P_m(x)P_n(x)\rho(x)dx = 0, \qquad m \neq n. \tag{92}$$

Equation(92) define $P_n(x)$ uniquely up to an arbitrary factor of proportionality, the normalisation constant. Given a weight function, the polynomials $P_n(x)$ can be computed explicity using the equation (92). We will not discuss the computation here *dagger*, instead we will derive some interesting general properties of orthogonal polynomials. As mentioned above, the primary aim is to establish the completeness of such sequences of orthogonal polynomials on any finite interval. In order to do this, we will need the following Lemma.

Lemma 3.10.1. *Every uniformly convergent sequence of continuous functions is mean-square convergent on any interval* $\tilde{I}$*, with respect to any integrable positive weight function* ρ*, i.e.,* $\int_{\tilde{I}} \rho(x)dx < +\infty$*.*

This follows immediately from the following inequality,

$$\int_{\tilde{I}} [f_n(x) - f(x)]^2 \rho(x)dx \leq \max\left[[f_n(x) - f(x)]^2\right] \int_{\tilde{I}} \rho(x)dx, \tag{93}$$

which is valid whenever $\tilde{I}$ is any finite interval. Using Lemma(3.10.1), it is easy to prove the completeness of a sequence of orthogonal polynomials defined on a finite interval J, relative to any continuous integrable weight function $\rho(x)$ from the fundamental result, Weierstrass Approximation Theorem[13],[1].

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† In principle, it is the Gram-Schimdt orthogonalization process applied to the vectors $1, x, x^2, \cdots$. This process can be applied in any Euclidean vector space.

Theorem 3.10.2. ***Weierstrass Approximation Theorem.*** *Let $f(x)$ be any function continuous on a finite closed interval $a \leq x \leq b$, and let $\epsilon > 0$ be any positive number. Then there exists a polynomial $p(x)$, such that $|p(x)|f(x)| \leq \epsilon$, for all $x \in [a, b]$.*

From Theorem(3.10.2) and the inequality (93), we infer the following

Theorem 3.10.3. *Let $P_n(x)$, for $n = 0, 1, 2, 3, \cdots$, be a polynomial function of degree n. For a fixed interval $[a, b]$, let*

$$\int_a^b P_m(x)P_n(x)\rho(x)\ dx = 0, \qquad m \neq n,$$

where $\rho(x)$ is a continuous integrable positive weight function. Then the orthogonal polynomials $P_n(x)$ are complete on $L^2_{\rho(x)}[a, b]$.

Proof. Let us start with the assumption that, $p(x)$ be a n degree polynomial. We can always find a constant c_n such that $p(x) - c_n P_n(x)$ is an $n - 1$ degree polynomial. Thus, due to induction on n, we can express $p(x)$ as a finite linear combination of $P_0(x), P_1(x), \cdots, P_n(x)$. By the Weierstrass Approximation Theorem(3.10.2), we can approximate uniformly any continuous function arbitrarily closely by a suitable polynomial $p(x)$.

Using Lemma(3.10.1), we get that every continuous function can be approximated arbitrarily closely in the mean square by a linear combination of the P_k. Therefore, Theorem(3.8.7), we get what we wanted to establish. This completes the proof. □

The completeness of orthogonal polynomials, whose corresponding SL system are defined over a finite interval like Legendre polynomials or Chebyshev polynomials follows as a corollary, but in general, it is difficult to establish the completeness of orthogonal polynomials like the Hermite polynomials or the Laguerre polynomials, whose corresponding SL systems are defined over intervals which are not finite.

CHAPTER 4: DISCRETE APPROXIMATION OF CHEBYSHEV COEFFICIENTS : PSEUDOSPECTRAL METHODS

4.1 Introduction

In theoretical analysis, it is a classic problem to find a proper basis of function classes and decompose functions from that class to write it in terms of the basis elements. We have seen the concerned methodologies and proceedings during Chapter(1). In numerical analysis, the primary motivation to represent functions as a series of other elementary functions like sinusoids or polynomials, was to facilitate solving differential equations efficiently. As described in [5], the basic idea is to assume that the unknown function $u(x)$ can be approximated by a sum of $N+1$ "basis functions" $\phi_n(x)$:

$$u(x) \approx u_N(x) = \sum_{n=0}^{N} a_n \phi_n(x)$$

When this series is substituted into the differential equation $Lu = f(x)$, where L is the operator of the differential, the result is the so-called *residual function* defined by:

$$\mathcal{R}(x; a_0, a_1, \cdots, a_N) = Lu_N - f$$

Since in the ideal scenario, that is, for the exact solution, the residual function

$$\mathcal{R}(x; a_0, a_1, \cdots, a_N)$$

is zero, the challenge is to compute the series coefficients $\{a_n\}$ so that the residual function admits a minimized value. Hereby, we dive into the realm of *spectral methods*. Other popular methods like *finite element methods* are similar in philosophy to spectral algorithms but the major difference is that finite elements chop the interval in x into a number of sub-intervals, and then choose the $\phi_n(x)$ to be local functions, which are polynomials of fixed degree which are non-zero only over a couple of sub-intervals. In contrast, spectral methods use global basis functions, in which $\phi_n(x)$ is a polynomial (or trigonometric polynomial) of high degree which is non-zero, except at isolated points, over the entire computational domain. The main disadvantage of finite element methods is low accuracy (for a given number of degrees of freedom N) because each basis function is a polynomial of low degree. In spectral methods, the high order of the basis functions give high accuracy for a given N, further spectral methods are also memory-minimizing.

Although we have seen many types of basis functions, the best choice for 95% of all applications is an ordinary Fourier series or a Fourier series in disguise. By "disguise" we mean a change of variable which turns the sines and cosines of a Fourier series into different functions. The most important disguise is the one worn by the Chebyshev polynomials, which have the property :

$$T_n(\cos\theta) \equiv \cos n\theta. \tag{94}$$

Due to this favorable property, Chebyshev polynomials are therefore of high interest as an alternative to the monomial basis for representing polynomials and polynomial expansions. This in a way forwards the motivation to compute *Chebyshev coefficients* in a faster manner, without compromising accuracy. In methodologies implemented during the numerical experiment on Fourier series approximation we had considered the formula involving the appropriate inner product to compute the SL series coefficients, which was a continuous case approximation. Unlike that, here we will explore about *discrete approximation of SL series coefficients, in particular Chebyshev coefficients.*

4.1.1 Examples of Orthogonal polynomials

1. *Legendre's Equation* : $(1-x^2)u'' - 2xu' + \lambda u = 0$ for $x \in (-1,1)$. Here, the eigenvalues are of the form $\lambda = n(n+1)$ for each $n \in \mathbb{N}_0$ and the eigenfunctions are polynomials called *Legendre's Polynomials*. The weight function here is 1, and the eigenfunctions are orthogonal in $L^2(-1,1)$.

   Named after the French mathematician Adrien-Marie Legendre (1752-1833).

2. *Hermite's Equation* : $u'' - 2xu' + \lambda u = 0$ for $x \in \mathbb{R}$. Here, the eigenvalues are of the form $\lambda = 2n$ for each $n \in \mathbb{N}_0$ and the eigenfunctions are polynomials called *Hermite's Polynomials*. The weight function here is e^{-x^2}, and the eigenfunctions are orthogonal in $L^2_{e^{-x^2}}(\mathbb{R})$.

   Named after the French mathematician Charles Hermite (1822-1901).

3. *Laguerre's Equation* : $xu'' - (1-x)u' + \lambda u = 0$ for $x \in (0,\infty)$. Here, the eigenvalues are of the form $\lambda = n$ for each $n \in \mathbb{N}_0$ and the eigenfunctions are polynomials called *Laguerre's Polynomials*. The weight function here is e^{-x}, and the eigenfunctions are orthogonal in $L^2_{e^{-x}}(0,\infty)$.

   Named after the French mathematician Edmond Laguerre (1834-1886).

4. *Chebyshev's Equation* : $(1-x^2)u'' - xu' + \lambda u = 0$ for $x \in (-1,1)$. Here, the eigenvalues are of the form $\lambda = n^2$ for each $n \in \mathbb{N}_0$ and the eigenfunctions are polynomials called *Chebyshev's Polynomials*. The weight function here is $(1-x^2)^{-1/2}$, and the eigenfunctions are orthogonal in $L^2_{(1-x^2)^{-1/2}}(-1,1)$.

   Named after the Russian mathematician Pafnuty Chebyshev (1821-1894).

For more detailed theory and numerical knowledge concerning orthogonal polynomials, [2], [15], [9], [25] are some popularly preferred sources. The Chebyshev and Legendre polynomials belong to the family of 'Gegenbauer polynomials'. Out of the four options mentioned above, we will particularly explore the fourth one, namely, Chebyshev polynomials.

4.2 Discrete form of Chebyshev coefficients

Spectral methods fall into two broad categories, namely, the *interpolating* and *non-interpolating* methods. The *interpolating* or *pseudospectral* methods associate a grid of points with each basis set. The coefficients of a given function $f(x)$ are calculated by requiring that the truncated series expansion agrees with $f(x)$ at each point of the grid. Similarly, the coefficients of a non-interpolating or pseudospectral approximation to the solution of a differential equation are found by requiring that the residual function interpolate $f = 0$ so that

$$\mathcal{R}(x_i; a_0, a_1, \cdots, a_N), \qquad i = 0, 1, 2, \cdots, N.$$

4.2.1 Motivation

Continuing the discussion initiated in the last paragraph, we know that in simple words, the pseudospectral method demands that the differential equation to be solved be exactly satisfied at a set of points known as the *collocation* or *interpolation* points. Presumably for certain class of algorithms, where the residual function is made to vanish at an increasingly large number of discrete points, it will be smaller and smaller in the gaps between the collocation points so that $\mathcal{R} \approx x$ everywhere in the domain, and therefore $u_N(x)$ will converge to $u(x)$ as N increases. Methods falling under the hood of this clan of algorithms are also called *orthogonal collocation* or *method of selected points*.

The *non-interpolating* clan of algorithms includes Galerkin's method[5] and the Lanczos Tau-method[5]. They do not use any grid of interpolation points. Instead, the coefficients of a given function $f(x)$ are computed by multiplying $f(x)$ by a given basis function and then integrating (somewhat philosophically similar to the methodologies we had implemented during the Fourier series experiment). It is indeed very enticing to report the difference between these two algorithmic clans as "integration-type" versus "interpolation-type", but it would be very naive in reality. Several classical books on approximation theory have shown how one can use the properties of the basis functions (recurrence relations, trigonometric identities, etc.) to compute coefficients without explicitly performing any integrations. Even though the end product is identically the same as that obtained by integration, it is a bit little confusing to label a calculation as an "integration-type" spectral method when there is not even a single integral sign in sight. Therefore, we shall use the benign label of "non-interpolating". [5] in its later chapters shows that the accuracy of pseudospectral methods is only a little bit poorer than that of the non-interpolating kingdom, to the extent that practically they are too little to outweigh the much greater simplicity and computational efficiency of the pseudospectral algorithms [*as evidence* Table(1)].

Consequently, we shall emphasize interpolating methods or pseudospectral methods in the sections following. Due to the working principle of pseudospectral algorithms we get to compute the so-called *discrete form* of the coefficients involved while doing an SL series approximation using Chebyshev polynomials. We will call these coefficients *Chebyshev coefficients*. Here-on, a Chebyshev polynomial of degree n will be denoted as $T_n(x)$.

$f(x)$	Error in continuous case	Error in discrete case	parameter n
$signum(x)$	9.9469e-03	1.0777e-02	$n = 128$
$\|x\|$	1.9764e-07	4.1633e-07	$n = 2$
x	2.9506e-26	3.1465e-32	$n = 2$
x^2	1.6725e-26	3.5160e-32	$n = 4$
$\cos x$	1.7464e-23	9.5802e-31	$n = 16$
$\sin x$	9.4316e-23	1.1464e-30	$n = 16$

Table 1: $L^2_{\rho(x)}$-error in approximations by continuous-case (Gaussian Integration) coefficients & discrete-case (Pseudospectral Chebyshev) coefficients

4.2.2 Calculations & computations

In order to get a detailed understanding of the contents of this subsection, it is of importance that one has a proper knowledge of numerical methods like *polynomial interpolation, trigonometric interpolation and Gaussian integration.* [7] is a good source of the purpose. Neveretheless, we will go through important definitions and results whenever they appear to be unavoidable. In the rest of this section, we will discuss about the choice of interpolation points and methods for computing the interpolant. A note on terminology: we shall use "collocation points" and "interpolation points" interchangeably.

Definition 4.2.1. ***Interpolation*** *An 'interpolating' approximation to a function $f(x)$ is an expression $P_{N-1}(x)$, usually an ordinary or trigonometric polynomial, whose N degrees of freedom are determined by the requirement that the 'interpolant' agree with $f(x)$ at each of a set of N interpolation points :*

$$P_{N-1}(x_i) = f(x_i), \qquad i = 1, 2, 3, \cdots, N.$$

Definition (4.2.1) above says that we can retrieve a function (or approximate it in a domain) using its values at discrete points inside the concerned domain.

Definition 4.2.2. ***Lagrange Interpolation Formula*** *In theory, we can fit values of an unknown function at $N+1$ points by using a polynomial of degree N as :*

$$P_N(x) \approx \sum_{i=0}^{N} f(x_i) C_i(x)$$

where $C_i(x)$ are polynomials of degree N, called the 'Cardinal functions' and are defined as :

$$C_i(x) = \prod_{j=0, j\neq i}^{N} \frac{x - x_j}{x_i - x_j}.$$

The cardinal function representation given above is not efficient for computation, although it gives a proof-by-construction of the theorem which says that it is possible to fit an interpolating polynomial of any degree to retrieve any function. Although the interpolating points are often evenly spaced or uniformly spaced (surely this is the most obvious possibility), but no such restriction is inherent in the formula given in Definition(4.2.2); the formula is still valid even if the $\{x_i\}$s are unevenly spaced or out of numerical order.

Case-study. Note that, it seems plausible that if we distribute the interpolation points evenly over an interval $[a, b]$, then the error in $P_N(x)$ should tend to 0 as $N \to \infty$ for any given smooth function f. Interestingly at the turn of the century, Runge provided an counterexample to show that this is not true. In his famous example, he had considered the function $f(x) = \frac{1}{1+x^2}$ in the interval $[-5, 5]$. Runge established that for this function, interpolation with evenly spaced points converges only within the interval $|x| \leq 3.63$ and diverges for larger $|x|$. In particular, the 15^{th} degree polynomial does an excellent job of representing the function for $|x| \leq 3$; but as we use more and more points, the error gets worse and worse near the endpoints.

Inferences. This numerical result reflected the fact that the situation would not be hopeless if one is willing to consider an uneven grid of interpolation points. As a matter of fact, Runge had proved the middle of the interval was not the problem. The big errors were always occuring near the endpoints. This suggested that one should space the grid points relatively far apart near the middle of the interval where we are getting high accuracy anyway and increase the density of grid points as we approach the endpoints.

Checkpoint. From the inferences of the Runge's counterexample, the idea to choose an uneven interpolation grid became hopeful. But now the challenge was to answer *what distribution of points is best?* The answer was discovered through a couple of classic theorems.

Theorem 4.2.3. ***Cauchy Interpolation Error Theorem*** *Let $f(x)$ have at least $(N+1)$ derivatives on the interval of interest and let $P_N(x)$ be its Lagrangian interpolant of degree N. Then*

$$f(x) - P_N(x) = \frac{1}{[N+1]!} f^{(N+1)}(\xi) \prod_{i=0}^{N} (x - x_i)$$

for some ξ on the interval spanned by x and the interpolation points. The point ξ depends on the function being approximated, upon N, upon x, and upon the location of the interpolation points.

Theorem 4.2.4. ***Chebyshev Minimal Amplitutde Theorem*** *Of all polynomials of degree N with leading coefficient (coefficient of x^N) equal to 1, the unique polynomial which has the smallest maximum on $[-1, 1]$ is $T_N(x)/2^{N-1}$, i.e., the N-th Chebyshev polynomial divided by 2^{N-1}. In other words, all polynomials of the same degree and leading coefficient unity satisfy the inequality*

$$\max_{x \in [-1,1]} |P_N(x)| \geq \max_{x \in [-1,1]} \left| \frac{T_N(x)}{2^{N-1}} \right| = \frac{1}{2^{N-1}}.$$

Proof of both Theorem(4.2.3) and Theorem(4.2.4) can be found at [11].

A proper understanding of '*Cauchy Interpolation Error Theorem*'(4.2.3) implies that, in order to optimize any Lagrangian interpolation, there is nothing one can do about the $f^{(N+1)}(\xi)$ factor ("in general") as it depends on the specific function being approximated, although the magnitude of the polynomial factor depends upon the choice of grid points. It is evident that the coefficient of x^N is 1, which is independent of the grid points, so the question becomes: *What choice of grid points gives us a polynomial (with leading coefficient 1), which is as small as possible over the interval spanned by the grid points?* It is true that by a linear change of variable, one can always rescale and shift any interval $[a, b]$ to $[-1, 1]$, but what after that? This impels us to seek help from the '*Chebyshev Minimum Amplitude Theorem*'(4.2.4). We use the fact that any polynomial of degree N can be factored into the product of linear factors of the form of $(x - x_i)$ where x_i is one of the roots of the polynomial, so in particular we have the following relation

$$\frac{1}{2^N} T_{N+1}(x) = \prod_{i=1}^{N+1} (x - x_i).$$

Thus, in order to minimize the error in the '*Cauchy Interpolation Error Theorem*'(4.2.3), the polynomial part of the remainder should be proportional to $T_{N+1}(x)$. *This implies that the optimum interpolation points are the roots of the Chebyshev polynomial of degree* $(N + 1)$. Invoking property (94), we know that Chebyshev polynomials are just cosine functions in disguise, hence these roots are given by

$$x_i = -\cos\left(\frac{(2i+1)\pi}{2N}\right), \qquad i = 0, 1, 2, \cdots, N. \tag{95}$$

$$OR$$

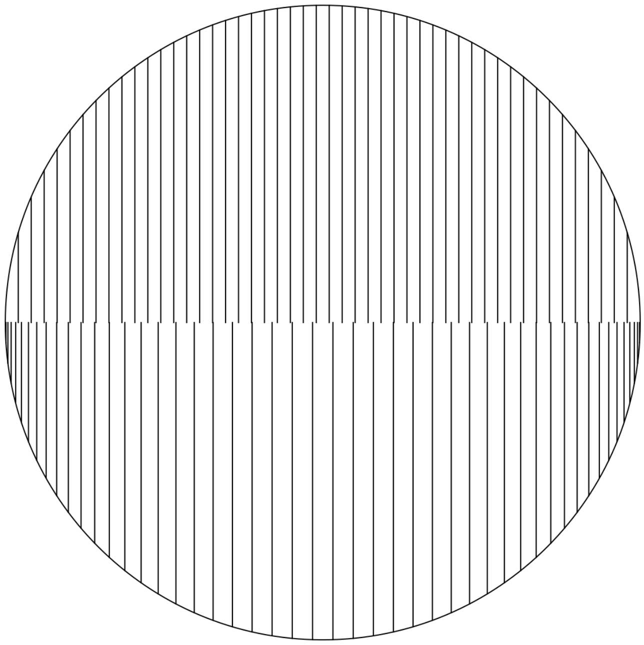
$$x_i = -\cos\left(\frac{i\pi}{2N}\right), \qquad i = 0, 1, 2, \cdots, N. \tag{96}$$

Analogous to Definition(4.2.2), if the interpolant $P_N(x)$ is of the form $\Sigma a_n T_n(x)$, then let us call the corresponding interpolation as the *Chebyshev Interpolation*. Further, the interpolation grid given by points in (95), which exclude the endpoints are called the *Chebyshev-roots grid*; whereas the grid given by points in (96), which includes the end-points are called *Chebyshev-extrema grid* or the 'Gauss-Lobatto grid'. We now borrow the definition and concept of the *discrete inner product* corresponding to regular inner product in $L^2[a, b]$ from [5], and formally state the following definition.

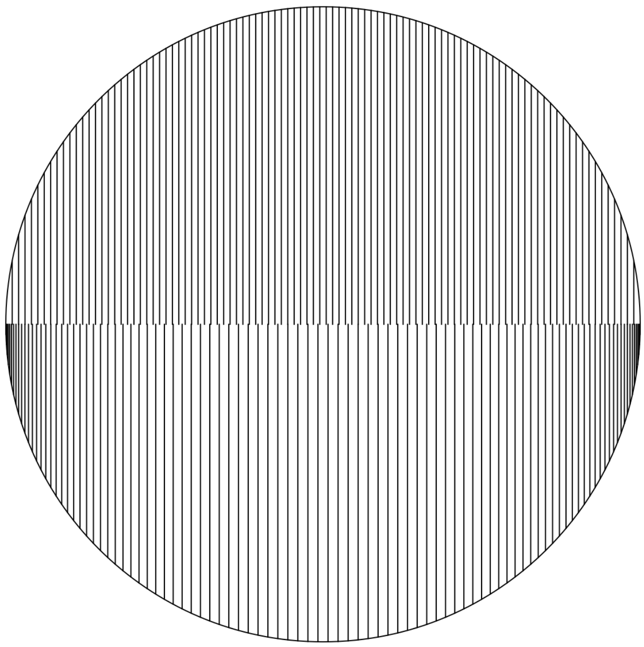
Note that, using the Taylor series expansion of cosine function we get that, near the endpoint $x = -1$, we have the grid points

$$x_1 \approx -1 + \frac{\pi^2}{8N^2}, \qquad x_2 \approx -1 + \frac{9\pi}{8N^2} \qquad [N >> 1].$$

Thus, these non-uniform grid points are at a spacing of $\mathcal{O}(1/N^2)$ near the end-points, where as the uniform grid points had a spacing of $\mathcal{O}(1/N)$. `[Ref. Figure(13)]`



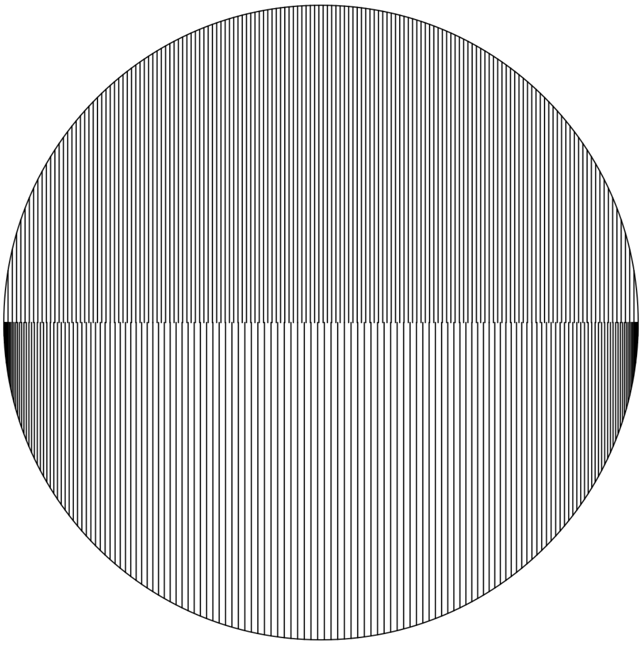
(a) 50 grid points.



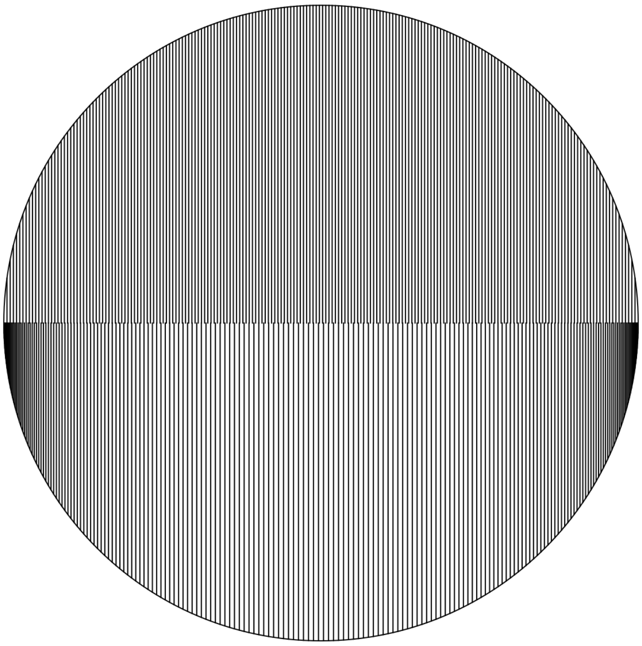
(b) 100 grid points.

Figure 13: Equidistant Grid (upper-half) vs. Chebyshev Extreme Grid (lower-half) in the interval $(-1, 1)$.

Note. In the above Figure(13), one can visualize the increase in density of grid points of Chebyshev Extreme Grid as compared to Equidistant Grid, near the end-points.



(a) 150 grid points.



(b) 200 grid points.

Figure 14: Equidistant Grid (upper-half) vs. Chebyshev Extreme Grid (lower-half) in the interval $(-1, 1)$.

Note. In the above Figure(14), one can visualize the increase in density of grid points of Chebyshev Extreme Grid as compared to Equidistant Grid, near the end-points.

Definition 4.2.5. ***Chebyshev Interpolation*** *If the N degree polynomial $P_N(x)$, which interpolates to a given function $f(x)$ at Chebyshev-extrema grid points (96) be defined as*

$$P_N(x) = \sum_{n=0}^{N}{}'' b_n T_n(x),$$

where the $''$ on the summation means that the first and last terms are to be taken with a factor of $1/2$, then the coefficients of the interpolating polynomial are given by

$$b_n = \frac{2}{N} \sum_{n=0}^{N}{}'' f(x_k) T_n(x_k) \qquad \text{[Extrema grid].} \tag{97}$$

If the N degree polynomial $Q_N(x)$, which interpolates to a given function $f(x)$ at Chebyshev-roots grid points (95) be defined as

$$Q_N(x) = \sum_{n=0}^{N}{}' c_n T_n(x),$$

where the $'$ on the summation means that the first $[c_0 T_0]$ is to be divided by $1/2$, then the coefficients of the interpolating polynomial are given by

$$c_n = \frac{2}{N+1} \sum_{n=0}^{N} f(x_k) T_n(x_k) \qquad \text{[Roots grid].} \tag{98}$$

Further, we say that the Chebyshev coefficients b_n at (97) is the Trapezoidal rule, and c_n at (98) is the Rectangular or Midpoint rule of denoting the Chebyshev coefficients.

There is a well explainable reason behind the trapeziodal and midpoint rule nomenclature of formulas (97) and (98). Recall that using the appropriate inner product, we already have Chebyshev coefficients a_n for a given function f from an appropriate function class as

$$a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) \cdot T_n(x)}{\sqrt{1-x^2}} dx$$

Here, after performing the substitution $x = \cos(\theta)$, and using (94) we get the following formula:

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(\cos\theta) \cdot T_n(\cos\theta)\ d\theta = \frac{2}{\pi} \int_0^{\pi} f(\cos\theta) \cdot \cos n\theta\ d\theta. \tag{99}$$

In (99), for periodic functions $f(x)$, the integral when computed with the *compact trapezoidal method of integration*[7] fetches us the exact formula given by (97), and similarly we get the formula (98) when we perform the *compact midpoint rule of integration*[7] to compute (99). Hence, this equivalence makes this nomenclature sensible.

Checkpoint. The next set of questions that immediately emerge after the introduction of this Chebyshev interpolation is *how accurate is Chebyshev interpolation? Does it converge over as wide a region as the usual Chebyshev expansion in which we compute a polynomial approximation by integration instead of interpolation?* From [7], and also from literature in general, we know that *Gaussian Integration* is the best method to compute integrals out of all known methods. One one hand, it provides results with least error, on the other it is computationally expensive too. The Trapezoidal Rule of integration is a very crude approximation of inegrals with a relative accuracy of only $\mathcal{O}(h^2)$ for general, i.e., for non-periodic functions. For periodic $f(x)$, however, the Trapezoidal Rule is equivalent to Gaussian integration. The following theorem, borrowed from [5] provides some clarity and reasoning on selecting the Trapezoidal rule and the Midpoint rule over Gaussian integration.

Theorem 4.2.6. ***Periodic Gaussian Quadrature*** *The Composite Trapezoidal Rule and Composite Midpoint Rule, situationally, are both a Gaussian quadrature in the sense that these formulas are exact with N points for trigonometric polynomials of degree $2N-2$.*

Note that the usage of the terminology "a Gaussian" quadrature is typically vague but necessary because there are actually two useful Gaussian quadratures associated with each basis. In the Fourier case, one can use either the Trapezoidal Rule or the Midpoint Rule; which are equally accurate. The Trapezoidal Rule, as used in the above, is the default choice, but the Midpoint Rule (or Rectangle Rule), which does not include either endpoints $x = 0$ or $x = 2\pi$ as grid points, is convenient when solving differential equations which have singularities at either of these endpoints.

4.3 Numerical Experiment

We consider different functions on the interval I and then conduct numerical experiments to approximate a given function f using discrete Chebyshev coefficients (midpoint rule), so that we could comment on the order of convergence of Chebyshev approximation. Our main aim is to numerically validate important theoretical results concerning the variance of order convergence of Chebyshev coefficients along with the function class from which functions are being approximated. The experiments were conducted in a system with *machine epsilon*, $\varepsilon = 2.220446049250313e - 16$.

Just to recall and for the sake of a mention, the code chunk in Listing(1) in principle computes discrete Chebyshev coefficients using the following formula

$$\boxed{c_n = \frac{2}{N}\sum_{i=0}^{N-1} f(\cos\frac{(2i+1)\pi}{2N})\cos\frac{(2i+1)n\pi}{2N} \qquad n = 0, 1, \cdots, N-1.}$$

4.3.1 Fast computation of polynomials & Clenshaw Algorithm

Now that a good approximation of Chebyshev coefficients has already been fixed, the next thing to focus on are methodologies implying fast evaluation of polynomials, particularly in our case, a truncated Chebyshev expansion or a *Chebyshev polynomials.* In general, let $\mathcal{P}_n$ denote the space of all real algebraic polynomials up to degree $n \in \mathbb{N}_0$,

$$p(x) := p_0 + p_1 x + \cdots + p_n x_n, \qquad x \in [a, b],$$

where $[a, b] \subset \mathbb{R}$ is a compact interval. [21], [7] discussed in detail the theory of *Horner's Scheme* of fast evaluation of polynomials. Its purpose is to compute any arbitrary polynomial, as above, with real coefficients p_k, $k = 0, \cdots, n$, at a certain point $x_0 \in [a, b]$ by a low number of arithmetic operations. The main idea behind the algorithm is to write $p(x_0)$ in the form of nested multiplications in order to reduce the number of needed multiplications, as follows

$$p(x_0) = p_0 + x_0\Bigg(p_1 + x_0\Big(p_2 + x_0\big(\cdots(p_{n-1} + x_0 p_n)\cdots\big)\Big)\Bigg).$$

All of this leads us to the popular Horner Scheme, Algorithm(2).

Evidently, the monomials x_k, $k = 0, \cdots, n$, form a simple basis of $\mathcal{P}_n$. Unfortunately, from a numerical point of view the monomial basis is undesirable. Thus, we are interested in another basis of $\mathcal{P}_n$ which is more convenient for numerical calculations. Using the Chebyshev polynomials, such a basis of $\mathcal{P}_n$ can be formed by the polynomials

$$T_k^{[a,b]}(x) := T_k\left(\frac{2x - a - b}{b - a}\right), \qquad k = 0, \cdots, n.$$

Hence, using this formula we obtain the following *shifted Chebyshev polynomials*

$$T_k^{[0,1]} := T_k(2x - 1).$$

Recall that, the following *three-term recurrence relation* is satisfied by Chebyshev polynomials

$$T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}(x), \qquad n \in \mathbb{N}. \tag{100}$$

Inspired from the Horner scheme, we iteratively try to reduce the degree of $p(x)$ by means of the recursion formula (100). Assume that $n \geq 5$ and $c_n \neq 0$. Applying (100) to T_n in the form of Q_N in Definiton(4.2.5), we obtain

$$p(x_0) = \frac{1}{2}a_0 + \sum_{k=1}^{n-3} a_k T_k(x_0) + (a_{n-2} - b_n)T_{n-2}(x_0) + b_{n-1}T_{n-1}(x_0)$$

with $b_n := a_n$ and $b_{n-1} := 2x_0 b_n + a_{n-1}$. Again applying (100), we can continue recursively, and hence can conclude that this *Cleshaw Algorithm*, Algorithm(1) is an analogon of *Horner Scheme*, Algorithm(2).

Algorithm 1: Clenshaw Algorithm

Data: $n \in \mathbb{N} \setminus \{1\},\ x_0 \in I,\ a_k \in \mathbb{R}$ for $k = 0, 1, \cdots, n$.
Result: $p(x_0) \in \mathbb{R}$.
Set $b_{n+2} = b_{n+1} := 0$ and calculate recursively;
while $j = 0, 1, ..., n$ **do**
 $b_{n-j} := 2x_0 b_{n-j+1} - b_{n-j+2} + a_{n-j}$.
end
Form $p(x_0) := \frac{1}{2}(b_0 - b_2)$.
Computational Cost: $\mathcal{O}(n)$.

4.3.2 Implementations: code

Here are the crucial code chunks on which the numerical experiments are mostly based, implementations of Chebyshev coefficient calculation with the roots grid and the Clenshaw Algorithm.

```python
# Python code chunk used in the numerical experiment...

def mapper(x, min_x, max_x, min_to, max_to):
    return (x - min_x) / (max_x - min_x) * (max_to - min_to) + min_to

def cheb_coef(func, n, min, max):
    coef = [0.0] * n
    for i in range(n):
        f = func(mapper(math.cos(math.pi * (i+0.5)/n), -1, 1, min, max)
    ) * 2/n
        for j in range(n):
            coef[j] += f * math.cos(math.pi * j * (i+0.5)/n)
    return coef
```

Listing 1: Computing Chebyshev coefficients using composite midpoint rule (97).

```python
# Python code chunk used in the numerical experiment...

def cheb_approx(x, n, min_, max_, coef):
    a = 1
    b = mapper(x, min_, max_, -1, 1)
    c = float('nan')
    res = coef[0] / 2 + coef[1] * b

    x = 2 * b
    i = 2
    while i < n:
        c = x * b - a
        res = res + coef[i] * c
        (a, b) = (b, c)
        i += 1

    return res
```

Listing 2: Code chunk for Clenshaw Algorithm.

4.4 Chebyshev series: convergence & errors

The impending problem associated with approximation is : *Given* $f \in C^k[a,b]$, *(be it periodic or non-periodic) what is the rate with which error of the best approximation, i.e,*

$$\inf_{p_n \in \mathcal{P}_n} \|f - p_n\|_\infty$$

converges to zero as $n \to \infty$ *?*

The answer in general is credited to Dunham Jackson, for the so called Jackson's Theorem(4.4.2) which is a classic theorem in Approximation Theory[4],[18]. It shows that the decay rate of the error depends on the extent of smoothness of the given function f. Later, it was due to Bernstein that the answer became sharper.

Theorem 4.4.1. ***Jackson's Theorem for 2π periodic functions*** *Let k run over $\mathbb{N}$ and $f \in C^k_{2\pi}[a,b]$. Then*

$$\inf_{p_n \in \mathcal{T}_n} \|f - p_n\|_\infty \leq \left(\frac{\pi}{2(n+1)}\right)^k \|f^{(k)}\|_\infty.$$

Theorem 4.4.2. ***Jackson's Theorem*** *Let n, k be integers with $n \geq k-1 \geq 0$ and $f \in C^k[-1,1]$. Then*

$$\inf_{p_n \in \mathcal{P}_n} \|f - p_n\|_\infty \leq \left(\frac{\pi}{2}\right)^k \frac{1}{(n+1)\cdots(n-k+2)} \|f^{(k)}\|_\infty.$$

In the first chapter, we had seen that for a sufficiently smooth 2π-periodic function f, its Fourier series converges uniformly and absolutely to itself on $\mathbb{T}$. Since Chebyshev series is a Fourier cosine series in disguise, this intuitively clarifies the absolute and uniform convergence of the Chebyshev series of a sufficiently smooth function f to itself on I.

Theorem 4.4.3. ***Bernstein*** *For $\mathbb{T} = [-\pi, \pi]$, let $f \in C^r(\mathbb{T})$ with fixed $r \in \mathbb{N}$ be given. Then the approximation error $f - S^f_n$ can be estimated for all $n \in \mathbb{N} \setminus \{1\}$ by*

$$\|f - s^f_n\| \leq c\|f^{(r)}\|_{C(\mathbb{T})} \frac{\ln r}{n^r},$$

where the constant $c > 0$ is independent of f and n.

In practice, the above convergence result(4.4.3) of a Fourier series is for a sufficiently smooth, 2π-periodic function is very useful. The proof can be found at [21]. Now if a given function $f \in C^r(I)$ with $r \in \mathbb{N}$, then the even function $\varphi = f(\cos \cdot)$ is contained in $C^r(\mathbb{T})$. By Theorem(4.4.3), we have

$$\lim_{n \to \infty} n^{r-1} \|\varphi - s^\varphi_n\|_{C(\mathbb{T})} = 0.$$

Using the result[21], $\varphi - s^\varphi_n = f - c^f_n$, we can conclude this discussion and present it as the following theorem.

Theorem 4.4.4. *For $f \in C^1(I)$, its corresponding Chebyshev series converges absolutely and uniformly on I to f. If $f \in C^r(I)$, for $r \in \mathbb{N}$, then we have*

$$\lim_{n \to \infty} n^{r-1} \|f - c^f_n\|_{C(I)} = 0,$$

where c^f_n is the corresponding Chebyshev partial sum of f.

Theorem(4.4.4) says about the uniform convergence of Chebyshev series, which here is important in the context as this enables us to change the metric to compute error in approximation. Previously, we had considered the appropriate L^2-norm of the difference in approximation as the error, but now we can conveniently use the discrete l_∞-norm for computing the error in approximation, which in return should facilitate a proper calculation of the order of convergence.

Using the above Theorem(4.4.4), we get that if a given function $f \in C^{r+1}(I)$ for a certain $r \in \mathbb{N}$, then for any given $\epsilon > 0$, $\exists\ n_0 \in \mathbb{N}\ \ni\ \forall n \geq n_0$ we have

$$\|f - c_n^f\|_{C(I)} < \frac{\epsilon}{n^r}.$$

In [19], the above result has been presented in the form of the following theorem.

Theorem 4.4.5. *If the function $f(x)$ has $m+1$ continuous derivatives on I , then $|f(x) - c_n^f(x)| = \mathcal{O}(n^{-m})$ for all $x \in I$.*

Qualitatively, we can comment that by virtue of Theorem(4.4.4), if the error in Chebyshev approximation $\|f - c_n^f\|_{C(I)}$ is denoted as E_n, then the extent of smoothness of the sample function directly affects the speed at which the approximation converges. Basically, *the smoother the function, the faster the approximation process converges to a minimum plausible approximation error E_n. In particular, the smoother a function $f : I \to \mathbb{R}$, the faster its Chebyshev series converges uniformly to f.* We can sharpen the above conclusion by re-mentioning a result, i.e, *theoretically* the Chebyshev approximation done using coefficients generated by Chebyshev roots grid(97), converges exponentially fast for periodic smooth functions. Note that, across all the numerical results from Table(4) to Table(15), the phenomenon of gradual drop in error of Chebyshev approximation E_n with increasing n for each of the corresponding functions can be answered by Theorem(4.4.4).

Numerically, these can be seen through the results presented in Section(3.6). Observe that for functions $x^m|x| \in C^m(I)$, an e^{-03} order accuracy is obtained for Chebyshev approximation with 128 terms for the function $|x|$. On the other hand, the same order accuracy has been achieved by only 8 terms of Chebyshev series for the function $x^7|x|$; not to mention that the Chebyshev series of $x^7|x|$ with 128 terms gives an e^{-14} order accuracy.

Theorem 4.4.6. ***Last-Coefficient Error Bound Theorem*** *The truncation error has the same order-of-magnitude as the last coefficient retained in the truncation for Chebyshev series, that is,*

$$E_n \sim \mathcal{O}(|a_n|).$$

In principle, the ultimate trial of any numerical solution is to repeat the relevant calculations with different N and compare the corresponding results. The above Theorem(4.4.6) is not intended to be a substitute for that[5]. Instead, it has two major purposes when seen as a 'rule-of thumb' for numerical experiments. First of all, it provides a quick-and-untidy way of estimating the approximation error E_n from a single calculation, that is, if the last retained coefficient is not small in comparison to the desired approximation error, then we need larger N . If it is small, and the lower coefficients decrease smoothly towards a_n, then the calculation is most probably alright. Secondly, it provides order-of-magnitude guidelines for estimating the feasibility of a calculation. Without such guidelines, one would waste lots of time by attempting problems which far exceed the available computing power. Quantitatively, Theorem(4.4.6) can be validated through the entries in Experiment Set A.

Definition 4.4.7. ***Order of Convergence for*** $\mathbf{C^k(I)}$ *For an algorithm or an iterative process, we define a constant called the order of convergence* μ *such that for the error at the* n^{th} *step the following holds*

$$|E_n| \sim \frac{1}{n^\mu}.$$

The Jackson's Theorem(4.4.2),(4.4.5) and Bernstien's Theorem(4.4.3), we can clearly anticipate the order of convergence for a given function f with known function class. In our context of all numerical experiments in Experiment Set A, analytically we have the following information, which is verified through the results data in the next section :

$f(x)$	k for $C^k(I)$	μ
$x^3 \sin 1/x$	1	–
$x^5 \sin 1/x$	2	1
$x^7 \sin 1/x$	3	2
$x^9 \sin 1/x$	4	3
$x^{11} \sin 1/x$	5	4
$x^{13} \sin 1/x$	6	5

Table 2: Actual data about input functions of Chebyshev Approximation Experiment Set A.

$f(x)$	k for $C^k(I)$	m for $C_p^m(I)$	μ
$\|x\|$	0	1	–
$x\|x\|$	1	2	2
$x^2\|x\|$	2	3	3
$x^3\|x\|$	3	4	4
$x^6\|x\|$	6	7	7
$x^7\|x\|$	7	8	8

Table 3: Actual data about input functions of Chebyshev Approximation Experiment Set B.

The reason for us to not being able to anticipate the order of convergence, μ for the instances mentioned in the first row of Table(2) and Table(3) lies in the hypothesis of Theorem(4.4.5). Clearly, the theorem only says about functions from $C^2(I)$ and its subspaces and it does not gaurantee anything about functions from $C^1(I)$ and $C^0(I)$. This also becomes a selling point of the numerical experiments we perfom, as in the cases when a result cannot be determined by theoretical analysis numerical analysis may determine it, although without a proof!

4.5 Some more results

Similarly as done in Section(1.14) for Fourier series, in this section we discuss some significant results related to the Chebyshev series of a function without going through their detailed proofs. significant topics discussed includes Minimax Approximation property of T_n, observation of Gibbs Phenomenon in Chebyshev Approximation, and Order of convergence μ for $C_p^k(I)$ functions, i.e, piecewise smooth functions.

4.5.1 Minimax Approximation Property

One area above all in which the Chebyshev polynomials T_n have a pivotal role is the *minimax approximation* of functions by polynomials. Hence, we will begin by reviewing some basic concepts of Approximation Theory before stating the main result. The main principle of approximation theory is to be able to replace any given function f by much a simpler form, such as a polynomial (mostly truncated expansion representations), chosen to have values not necessarily identical with but very close to those of the given function, since such an 'approximation' may not only be more compact to represent and store the given function computationally, but also more efficient to evaluate or otherwise manipulate.

Definition 4.5.1. ***Approximation Space*** *For a known function class $\mathcal{F}$ and a given function $f \in \mathcal{F}$ to be approximated, we define a family $\mathcal{A}$ of all possible approximations $f^*(x)$ to the given function $f(x)$.*

Example. We might choose our approximation from one of the following families:

1. Polynomials of degree n, where
$$\mathcal{A} = \{f^*(x) = p_n(x) = c_0 + c_1 x + \cdots + c_n x^n\} \quad \text{(parameters } c_j\text{)}.$$
2. Rational functions of type (p, q), where
$$\mathcal{A} = \left\{ f^*(x) = r_{p,q}(x) = \frac{a_0 + a_1 x + \cdots + a_p x^p}{1 + b_1 x + \cdots + b_q x^q} \right\} \quad \text{(parameters } a_j,\ b_j\text{)}.$$
3. Trigonometric Polynomials of degree $2n$, where
$$\mathcal{A} = \{f^*(x) = t_n(x) = a_0 + a_1 \sin x + \cdots + a_n \sin nx + b_1 \cos x + \cdots + b_n \cos nx\}$$
$$\text{(parameters } a_j,\ b_j\text{)}.$$

Now we can mention a rule-of-thumb for numerical puroses. $\mathcal{A}$ also forms a function space. The rule-of-thumb is to choose the approximation space such that $\mathcal{A} \subset \mathcal{F}$, where the given function $f \in \mathcal{F}$. In contrast to $\mathcal{F}$, $\mathcal{A}$ is a finite dimensional function space, its dimension being the number of parameters in the form of approximation and Parseval's Identity of Chebyshev series.

Now that we have defined *approximation space*, we are in a good shape to define and state what usually is called an *approximation problem.* The definiton arises from the need to check the quality of approximation to be done for a given function $f(x)$. In practice there are three types of approximation that are commonly aimed for.

Definition 4.5.2. *Let $\mathcal{F}$ be a normed linear space, let $f(x)$ in $\mathcal{F}$ be given, and let $\mathcal{A}$ be a given subspace of $\mathcal{F}$.*

1. *An approximation $f^*(x)$ in $\mathcal{A}$ is said to be good (or acceptable) if*
$$\|f - f^*\| < \epsilon$$
for any given $\epsilon > 0$. ϵ denotes the desired level of absolute accuracy.

2. *An approximation $f_B^*(x)$ in $\mathcal{A}$ is a best approximation if, for any other approximation $f^*(x)$ in $\mathcal{A}$, we have*

$$\|f - f_B^*\| \leq \|f - f^*\|.$$

Note that there will sometimes be more than one best approximation to the same function.

Recall that, in the case Fourier series Theorem(1.14.5), that is, the Weierstrass Approximation Theorem for trigonometric polynomials says about the approximation problem of the first type mentioned in Definition(4.5.2), with $\mathcal{A}$ as mentioned in the third example above. Also, the least square property(26) as mentioned in Section(1.14) of Chapter(1) basically discusses an approximation problem of the second type, with same $\mathcal{A}$.

Further, for a given norm $\|\cdot\|$ (such as $\|\cdot\|_\infty$, $\|\cdot\|_2$ or $\|\cdot\|_1$), a best approximation of the second type as defined in Definition(4.5.2), is a solution of the problem

$$\underset{f^* \in \mathcal{A}}{minimize} \, \|f - f^*\|.$$

In the case of polynomial approximation, to which we now restrict our attention, we may rewrite the above expression in terms of the parameters as

$$\underset{c_0, \cdots, c_n}{minimize} \, \|f - p_n\|.$$

Note that the Weierstrass Approximation Theorem[13] guarantees in particular the existence of a unique best approximation in the $\mathcal{L}_\infty$ or *minimax norm*. The best $\mathcal{L}_\infty$ or minimax approximation problem is (in concise notation)

$$\underset{c_0, \cdots, c_n}{minimize} \max_{a \leq x \leq b} |f(x) - p_n(x)|.$$

The Weierstrass Approximation Theorem does provide an existence of a best polynomial approximation, but it can be sharpened in the form of the following powerful theorem, which we borrow from [19].

Theorem 4.5.3. ***Alternation theorem for polynomials*** *For any $f(x)$ in $C[a, b]$ a unique minimax polynomial approximation $p_n(x)$ exists, and is uniquely characterised by the 'alternating property' that there are $n+2$ points (at least) in $[a, b]$ at which $f(x) - p_n(x)$ attains its maximum absolute value (namely $\|f - p_n\|_\infty$) with alternating signs.*

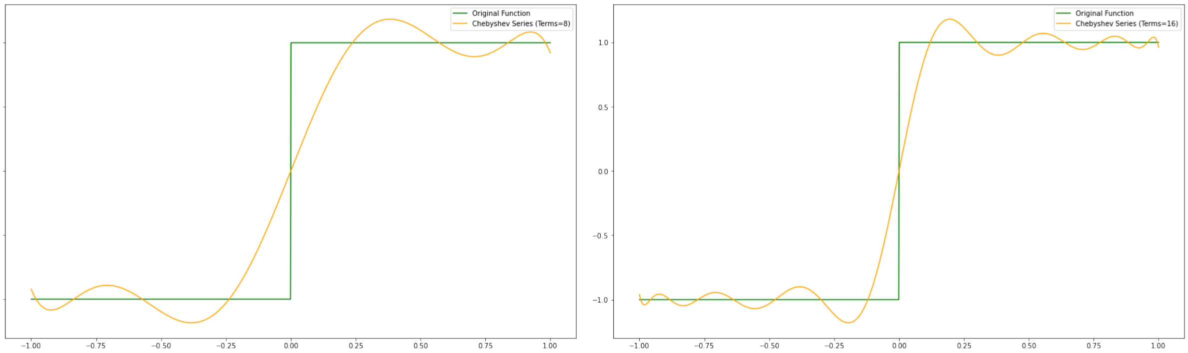
Theorem(4.5.3), often ascribed to Chebyshev but more properly attributed to Borel, asserts that, for p_n to be the best approximation, it is both necessary and sufficient that the alternating property should hold, that only one polynomial has this property, and that there is only one best approximation. Note that similar to this, the Fourier series solves the approximation problem in accordance to the celebrated theorem named the *Alternation theorem for trigonometric polynomials.*

We have the following result as a corollary[19] to Theorem(4.5.3), which answers that a truncated Chebyshev series, as a polynomial approximation solves the *minimax polynomial approximation problem.* Its interpretation shares genesis from Theorem(4.2.4), stated above.

Corollary 4.5.4. *The minimax polynomial approximation of degree* $n-1$ *to the function* $f(x) = x^n$ *on* I *is*

$$p_{n-1}(x) = x^n - 2^{1-n}T_n(x).$$

Further, $2^{1-n}T_n(x)$ *is the minimax approximation on* I *to the zero function by a monic polynomial of degree* n.



(a) Plot for the signum function and its Chebyshev partial sum c_8^f.

(b) Plot for the signum function and its Chebyshev partial sum c_{16}^f.

Figure 15: Plot for functions with sharp points & discontinuity and their respective Chebyshev partial sum [Discrete Chebyshev coefficients].

Note that not only the Sturm Oscillation Theory discussed in Chapter(2), Theorem(4.5.3) can also be aided and made intuitively clear with the help of the following lemma. Its effect can be seen in the Chebyshev Approximation of the signum function as shown in Figure(15). We can see that in both the plots, the truncated Chebyshev expansion coincides with the function exactly at the corresponding $n + 1$ points.

Lemma 4.5.5. ***Alternating property of*** $\mathbf{T_n(x)}$ *On* I, $T_n(x)$ *attains its maximum magnitude of 1 with alternating signs at precisely* $n + 1$ *points namely at the points*

$$x_k = \frac{\cos k\pi}{n}, \qquad k = 0, 1, \cdots, n.$$

Proof. Due to the strong relation (94), and the fact that $\cos n\theta$ attains its maximum magnitude of unity with alternating signs at its extrema points, we get what we wanted to proof. This completes the proof. $\square$

4.5.2 Order of convergence μ for $C_p^k(I)$

Although we have Theorem(4.4.5) for anticipating the order of convergence μ for $C^k(I)$ functions, but in this section we will mention and discuss some results which sharpens this theorem for $C_p^k(I)$ functions, i.e., piecewise k-times continuously differentiable functions.

We invoke the exact same computations done by us in Section(1.5) of Chapter(1). The methodology is to repeatedly perform integration by parts. For a convenient demonstration let us assume that the given function belongs to $C_p^2(I)$, i.e., the function is 2 times piecewise continuously differentiable, and since we are working in a compact interval let us assume that $\lambda_2 = \sup|f'(x)|\ \forall x \in I$. We have seen that the Chebyshev coefficient are of the following form due to (94)

$$
\begin{aligned}
c_n^f &= \int_0^\pi f(\cos x)\cdot \sin nx\ dx \\
&= \underbrace{-\frac{1}{n} f(\cos x)\cdot \sin nx\Big|_0^\pi}_{\text{vanishes}} - \frac{1}{n}\int_0^\pi f'(\cos x)\sin x\cdot \sin nx\ dx \\
&= \underbrace{-\frac{1}{n} f'(\cos x)\int_0^\pi \sin x\cdot \sin nx\ dx}_{\text{vanishes as the integral goes to 0}} + \underbrace{\frac{1}{n}\int_0^\pi f''(\cos x)\sin x\Big[\frac{\cos x\cdot \sin nx - n\sin x\cdot \cos nx}{n^2-1}\Big]\ dx}_{\mathcal{O}(\frac{1}{n^3})}
\end{aligned}
$$

Thus, using Theorem(4.4.6) we can present the implication of these calculation in the form of the following theorem.

Theorem 4.5.6. ***Order of Convergence for*** $\mathbf{C_p^k(I)}$ *If the given function $f \in C_p^k(I)$, then the error in approximation of the Chebyshev approximation is*

$$|E_n| \sim \mathcal{O}(|a_n|) \sim \mathcal{O}\frac{1}{n^k}.$$

Hence, the order of convergence for such functions will be k.

One can easily validate the above mentioned Theorem(4.5.6), through the numerical data presented under the Experiment Set B in the upcoming section, where functions considered piecewise smooth of some extent.

4.5.3 Occurence of Gibbs Phenomena

Note that, the Gibbs Phenomenon discussed in Section(1.14) for Fourier partial sums can also be seen qualitatively in both the plots of Figure(15) for Chebyshev partial sums of the signum function. It makes sense as a Chebyshev series is in principle a Fourier Cosine series. By ‘qualitatively’, we mean the uneven oscillations near both end-points.

4.6 Numerical results

4.6.1 Experiment Set A

n	$\|f - c_n^f\|_\infty$	$\|a_n\|$	μ_n
2	8.31e-01	2.18e-02	–
4	4.61e-01	2.23e-01	8.50e-01
8	4.11e-02	8.78e-02	3.49e+00
16	9.18e-11	1.98e-12	2.87e+01
32	2.63e-11	7.20e-13	1.80e+00
64	8.93e-13	5.52e-14	4.88e+00
128	1.19e-14	3.57e-16	6.23e+00

Table 4: $x^{13} \sin 1/x$.

n	$\|f - c_n^f\|_\infty$	$\|a_n\|$	μ_n
2	8.20e-01	4.37e-02	–
4	3.95e-01	2.61e-01	1.05e+00
8	2.21e-02	7.58e-02	4.16e+00
16	1.19e-08	3.67e-10	2.08e+01
32	2.39e-09	1.05e-10	2.32e+00
64	1.94e-11	3.50e-12	6.94e+00
128	1.03e-12	1.52e-14	4.23e+00

Table 5: $x^{11} \sin 1/x$.

n	$\|f - c_n^f\|_\infty$	$\|a_n\|$	μ_n
2	7.98e-01	8.73e-02	–
4	3.19e-01	3.06e-01	1.32e+00
8	7.81e-03	5.73e-02	5.35e+00
16	9.72e-07	4.77e-08	1.30e+01
32	1.01e-07	9.55e-09	3.27e+00
64	3.27e-09	5.85e-11	4.94e+00
128	1.49e-10	1.97e-12	4.46e+00

Table 6: $x^9 \sin 1/x$.

n	$\|f - c_n^f\|_\infty$	$\|a_n\|$	μ_n
2	7.54e-01	1.75e-01	–
4	2.29e-01	3.58e-01	1.72e+00
8	1.29e-04	3.12e-02	1.08e+01
16	3.99e-05	3.89e-06	1.69e+00
32	1.40e-06	4.02e-07	4.83e+00
64	1.81e-07	1.12e-08	2.96e+00
128	1.46e-08	4.55e-10	3.63e+00

Table 7: $x^7 \sin 1/x$.

n	$\|f - c_n^f\|_\infty$	$\|a_n\|$	μ_n
2	6.67e-01	3.49e-01	–
4	1.25e-01	4.17e-01	2.42e+00
8	3.88e-03	5.15e-04	5.01e+00
16	5.70e-04	1.60e-04	2.77e+00
32	1.13e-04	1.54e-06	2.34e+00
64	1.35e-05	6.87e-07	3.07e+00
128	2.72e-06	4.72e-08	2.31e+00
256	5.77e-07	2.36e-09	2.24e+00
512	9.60e-08	2.66e-10	2.59e+00

Table 8: $x^5 \sin 1/x$.

n	$\|f - c_n^f\|_\infty$	$\|a_n\|$	μ_n
2	4.62e-01	7.80e-02	–
4	4.34e-02	1.86e-01	3.41e+00
8	2.30e-02	2.69e-02	9.16e-01
16	1.03e-02	1.59e-03	1.17e+00
32	3.07e-03	1.63e-04	1.74e+00
64	7.95e-04	1.90e-04	1.95e+00
128	4.50e-04	4.06e-05	8.23e-01
256	1.55e-04	8.79e-06	1.54e+00
512	7.57e-05	1.54e-06	1.03e+00

Table 9: $x^3 \sin 1/x$.

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‡ All the numerical experiments done during the span of this project can be found at the GitHub Repo with the link - https://github.com/Shubhajit412/THEORETICAL-AND-COMPUTATIONAL-CONSIDERATIONS-OF-STURM-LIOUVILLE-SYSTEMS.

4.6.2 Experiment Set B

n	$\|f - c_n^f\|_\infty$	Order of Convergence
2	9.12e-01	–
4	3.07e-01	1.57e+00
8	2.84e-03	6.75e+00
16	1.16e-06	1.13e+01
32	3.06e-09	8.57e+00
64	1.09e-11	8.14e+00
128	4.11e-14	8.04e+00

Table 10: $x^7|x|$.

n	$\|f - c_n^f\|_\infty$	Order of Convergence
2	9.12e-01	–
4	3.07e-01	1.57e+00
8	3.62e-03	6.41e+00
16	4.87e-06	9.54e+00
32	2.90e-08	7.39e+00
64	2.12e-10	7.09e+00
128	1.62e-12	7.03e+00

Table 11: $x^6|x|$.

n	$\|f - c_n^f\|_\infty$	Order of Convergence
2	6.46e-01	–
4	5.97e-02	3.44e+00
8	1.12e-03	5.74e+00
16	5.49e-05	4.35e+00
32	3.26e-06	4.07e+00
64	2.01e-07	4.02e+00
128	1.25e-08	4.00e+00

Table 12: $x^3|x|$.

n	$\|f - c_n^f\|_\infty$	Order of Convergence
2	6.46e-01	–
4	9.57e-02	2.76e+00
8	5.72e-03	4.06e+00
16	6.27e-04	3.19e+00
32	7.60e-05	3.05e+00
64	9.40e-06	3.01e+00
128	1.16e-06	3.02e+00

Table 13: $x^2|x|$.

n	$\|f - c_n^f\|_\infty$	Order of Convergence
2	2.93e-01	–
4	3.81e-02	2.94e+00
8	7.58e-03	2.33e+00
16	1.78e-03	2.09e+00
32	4.39e-04	2.02e+00
64	1.09e-04	2.01e+00
128	2.72e-05	2.01e+00

Table 14: $x|x|$.

n	$\|f - c_n^f\|_\infty$	Order of Convergence
2	7.06e-01	–
4	2.70e-01	1.39e+00
8	1.26e-01	1.09e+00
16	6.18e-02	1.03e+00
32	3.03e-02	1.03e+00
64	1.46e-02	1.05e+00
128	6.84e-03	1.10e+00

Table 15: $|x|$.

‡ All the numerical experiments done during the span of this project can be found at the GitHub Repo with the link - https://github.com/Shubhajit412/THEORETICAL-AND-COMPUTATIONAL-CONSIDERATIONS-OF-STURM-LIOUVILLE-SYSTEMS.

Appendices

I Supplementary pertinent theorems and lemmas

I.I Some Comparison theorems on first-order DEs

It is a well-established fact that most DEs cannot be solved in terms of elementary functions, so it becomes important to be able to comapre unknown non-trivial solutions of one DE with some known non-trivial solution of another. In this section we present some useful comparison theorems on DEs, which vcan be used as efficient tools in solving relevant problems. Proofs of the following results are omitted here, but can be found at [2]-`page29`.

Definition I.I.I. *A function $F(x, y)$ is said to satisfy a* ***Lipschitz condition*** *in a domain D when, for some finite non-negative constant L (Lipschitz constant), the inequality*

$$|F(x, y_2) - F(x, y_1)| \leq L|y_2 - y_1|$$

holds for all pairs (x, y_2) and (x, y_1) in D having the same x-coordinate.

Lemma I.I.II. *Let F be continuously differentiable in a bounded closed convex domain D. Then it satisifes a Lipschitz condition there, with $L = sup_D \left|\frac{\partial F}{\partial y}\right|$.*

Theorem I.I.III. *We consider two DEs in the interval (a, b),*

$$y' = F(x, y), \qquad y_1' = G(x, y_1)$$

such that $F \leq G$. Let G satisfy a Lipschitz condition for $x \geq a$. If f and g are some non-trivial solutions of the first and second DEs above, respectively satisfying the initial condition $g(a) = f(a)$, then $f(x) \leq g(x)$ for all $x \geq a$.

Theorem I.I.IV. *Let f and g be two non-trivial solutions of the following DEs,*

$$y' = F(x, y), \qquad z' = G(x, z)$$

respectively, where $F(x, y) \leq G(x, y)$ in the closed interval $[a, b]$ and both F and G satisfy a Lipschitz condition. If the intial condition $f(a) = g(a)$ is satisfied, then $\forall x \in [a, b]$, we have $f(x) \leq g(x)$.

Theorem(I.I.III) can be sharpened in the form of the following corollary.

Corollary I.I.V. *In theorem(I.I.III), we assume that along with G, F too satisfies a Lipschitz condition and instead of $f(a) = g(a)$, we assume the intial condition $f(a) < g(a)$. Then $f(x) < g(x)$ for all $x > a$.*

Corollary I.I.VI. *In Theorem(I.I.IV), if additionally we have $f(b) = g(b)$, then $f(x) \equiv g(x)$ for all $x \in [a, b]$.*

I.II An alternate qualitative analysis on zeroes of eigenfunctions

Section(3.1) discusses things of the same genre as of the contents of this section. Here, we will go through similar results and try to present their proofs a bit differently. Well, by now it is evident and also it has been mentioned several times that DEs of the type

$$y'' + q(x)y' + r(x)y = 0, \qquad \forall x \in I, \tag{101}$$

explicitly so that one can study the properties of its solutions. We shall see ahead, how the coefficients $q(x)$ and $r(x)$ affect the way the zeroes of non-trivial solutions of DEs like (101) behave. We will use the concept of the Wronskian[2]-`page43`.

Suppose at a certain point x_0 in the interval I, we have $y(x_0) = 0$, where y is a solution of DE(101). If $y'(x_0) \neq 0$, as by uniqueness theorem it would imply y to be the trivial solution. Now as y' is continuous on I, there exists a neighborhood U of x_0 where $y' \neq 0$ on $U \cap I$. Thus, y is either strictly increasing or strictly decreasing on $U \cap I$. Using definiton(2.4.3), we can conclude that, *if y is a non-trivial solution of DEs like (101), then the zeroes of y are isolated in I.*

Now, say y_1 and y_2 are two non-trivial linearly independant solutions of (101). It follows that, their Wronskian

$$W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

does not vanish on the interval I, and thus has one sign on I. This also tells us that, both y_1 and y_2 cannoit have a common zero. Let us assmue that x_1 and x_2 are two consecutive zeroes of y_2. Then, it is clear that

$$W(x_1) = y_1(x_1)y_2'(x_1) \neq 0,$$
$$W(x_2) = y_1(x_2)y_2'(x_2) \neq 0.$$

Therefore, the values $y_1(x_1)$, $y_1(x_2)$, $y_2'(x_1)$ and $y_2'(x_2)$ are all non-zero. Since y_2' is continuous on I, x_1 has a neighborhood U_1 where the sign of y_2' does not change. Similarly for x_2 there is a neighborhood U_2 where y_2' does not change its sign. Continuity of y_2 says that if y_2 is increasing on a neighborhood of one root then it has to be decreasing on a neighborhood of the other, hence the signs of y_2' in $U_1 \cap I$ and $U_2 \cap I$ cannot be the same. Now, for $W(x)$ to have a constant sign on I, $y_1(x_1)$ and $y_1(x_2)$ must have opposite signs. Hence y_1, being continuous, has at least one zero between x_1 and x_2.

Now say, x_3 and x_4 are two zeroes of y_1 which lie between x_1 and x_2, we can employ similar arguments to show that y_2 vanishes between x_3 and x_4, which is a contradiction to the assumption that x_1 and x_2 are successive zeroes of y_2. Therefore, we can conclude that y_1 has exactly one zero between x_1 and x_2. This result is a well celebrated result in the Theory of Sturm-Liouville Systems and is often presented as the following theorem.

Theorem I.II.I. ***Strum Separation Theorem*** *If y_1 and y_2 are linearly independent solutions of the equation(101), then the zeros of y_1 are distinct from those of y_2, and the two sequences of zeros alternate. Precisely, y_1 has exactly one zero between any two successive zeros of y_2, and viceversa.*

Corollary I.II.II. *If two non-trivial solutions of DE(101) have a common zero in I, then they are linearly dependent.*

I.III Additional definitions & theorems

Definition I.III.I. *A family of vector fields $X(\mathbf{x}, t)$ satisfies a Lipschitz condition in a region $\mathcal{R}$ of $(\mathbf{x}, t)-$space if and only if, for some Lipschitz constant L,*

$$|X(\mathbf{x}, t) - X(\mathbf{y}, t)| \leq L|\mathbf{x} - \mathbf{y}| \quad \text{if} \quad (\mathbf{x}, t), (\mathbf{y}, t) \in \mathcal{R}$$

Theorem I.III.II. *Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ satisfy the DEs*

$$\frac{dx}{dt} = X(\mathbf{x}, t) \quad \text{and} \quad \frac{dy}{dt} = Y(\mathbf{y}, t)$$

respectively, on $a \leq t \leq b$. Further, let the functions X and Y be defined and continuous in a common domain $\mathcal{R} := D \times [a, b]$, and let

$$|X(\mathbf{z}, t) - Y(\mathbf{z}, t)| \leq \epsilon, \quad a \leq t \leq b, \quad \mathbf{z} \in D.$$

Finally, if $X(\mathbf{x}, t)$ satisfy the Lipschitz condition (I.III.I), then

$$|\mathbf{x}(t) - \mathbf{y}(t)| \leq |\mathbf{x}(a) - \mathbf{y}(a)|e^{L|t-a|} + \frac{\epsilon}{L}[e^{L|t-a|} - 1].$$

Note that in theorem(I.III.II), $\mathbf{Y}$ is not required to satisfy a Lipschitz condition. The proof of (I.III.I) can be found at [2].

Theorem I.III.III. *For fixed $r \in \mathbb{N}_0$, let $f \in C^{r+1}(I)$ be given. Then for all $n > r$, the Chebyshev coefficients of f satisfy the inequality*

$$\|c_n^f\| \leq \frac{2}{n(n-1)\cdots(n-r)}\|f^{r+1}\|_{C(I)}.$$

Proof of Theorem(I.III.III) is owed to [21], where it goes by Theorem 6.16 at page 319.

Theorem I.III.IV. ***Chebyshev Interpolation Error Bound*** *If the N degree polynomial $P_N(x)$, which interpolates to a given function $f(x)$ at Chebyshev-extrema grid points (96) be defined as*

$$P_N(x) = \sum_{n=0}^{N}{}'' b_n T_n(x),$$

where the $''$ on the summation means that the first and last terms are to be taken with a factor of $1/2$, then the coefficients of the interpolating polynomial are given by

$$b_n = \frac{2}{N}\sum_{n=0}^{N}{}'' f(x_k)T_n(x_k) \qquad \text{[Extrema grid]}.$$

If the N degree polynomial $Q_N(x)$, which interpolates to a given function $f(x)$ at Chebyshev-roots grid points (95) be defined as

$$Q_N(x) = \sum_{n=0}^{N}{}' c_n T_n(x),$$

where the $'$ on the summation means that the first $[c_0 T_0]$ is to be divided by $1/2$, then the coefficients of the interpolating polynomial are given by

$$c_n = \frac{2}{N+1} \sum_{n=0}^{N} f(x_k) T_n(x_k) \qquad \text{[Roots grid].}$$

Let $\{\alpha_n\}$ denote the exact spectral Chebyshev coefficients of $f(x)$. Then for all N and all real $x \in [-1, 1]$, the errors in either of the interpolating polynomials is bounded by twice the sum of the absolute values of all the neglected coefficients, basically

$$|f(x) - P_N(x)| \leq 2 \sum_{n=N+1}^{\infty} |\alpha_n|$$

$$|f(x) - Q_N(x)| \leq 2 \sum_{n=N+1}^{\infty} |\alpha_n|.$$

Proof of Theorem(I.III.IV) is owed to [5], where it goes by Theorem 21 at page 97.

Theorem I.III.V. *Let $f \in C^{r+1}(I)$ for fixed $r \in \mathbb{N}$ be given. Assume that $N \in \mathbb{N}$ with $N > r$. If the N degree polynomial $Q_N(x)$, which interpolates to a given function $f(x)$ at Chebyshev roots grid points exactly as mentioned in Lemma(4.2.5), then we have*

$$\|f(x) - Q_N(x)\|_{C(I)} \leq \frac{4}{r(n-r)^r} \|f^{r+1}\|_{C(I)}.$$

From [5], we borrow the fact that for a function f the actual spectral Chebyshev coefficients $\{\alpha_n^f\}$ are always less than the approximated Chebyshev coefficients $\{c_n^f\}$, that is, $\alpha_n^f < c_n^f$. As a consequence of Theorem(I.III.III) and Theorem(I.III.IV) we get the above Theorem(I.III.V).

II Supplementary algorithms

Algorithm 2: Horner's Scheme Algorithm

Data: $n \in \mathbb{N} \setminus \{1\},\ x_0 \in [a,b],\ p_k \in \mathbb{R}$ for $k = 0, 1, \cdots, n$.

Result: $p(x_0) \in \mathbb{R}$.

Set $q_{n-1} := p_n$ and calculate recursively;

while $j = 2, ..., n$ **do**

  $q_{n-j} := p_{n-j+1} + x_0 q_{n-j+1}$.

end

Form $p(x_0) := p_0 + x_0 q_0$.

Computational Cost: $\mathcal{O}(n)$.

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